Principles of Analysis

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To the Instructor

This book is based upon the one-semester transitional course *Principles of Analysis* taught at the College of the Holy Cross. Starting from axioms for the real numbers, the book develops the integral and differential calculus of one real variable, culminating with a construction of exponential and circular functions via integration and power series. Familiar properties of these functions are established from their characterizing differential equations.

The target readership is sophomore-level mathematics majors. The prerequisites are modest: a year of calculus, the ability to read actively, and a capacity for abstract reasoning.

Students can easily find their first exposure to analysis to be a blizzard of complicated and unfamiliar definitions, bewildering idioms, and esoteric-sounding consequences stated as tone poems written from Greek alphabet soup. Indeed, basic analysis is a *language* course as much as a mathematics course. When this fact is not explicitly acknowledged, students are unnecessarily burdened with a mysterious layer of frustration: Students who enjoyed calculus suddenly find their day-to-day understanding of mathematics has sharply decreased for no apparent reason.

Of course, the definitions of analysis exist to clarify potentially murky (albeit familiar) notions involving infinitesimals and the infinite, and theorems have useful computational and analytic applications. The book takes special care to articulate and establish technical points that are glossed over or omitted from modern calculus courses, and to assist students' absorption of analytic syntax and idioms by expressing conclusions of theorems in multiple forms: formally, conceptually/verbally, and in practical terms of their mathematical consequences.

The organization and expository style are in part responses to the modern calculus syllabus and the changing strengths of college students.

Following an old suggestion of Knuth, computational notation for estimates, a.k.a. interval arithmetic, is developed, and used where it seems to simplify "traditional" proofs.

Though fairly strict logical development is maintained (and minor exceptions are clearly indicated), the material is not always separated traditionally into topics. For example, integration by parts is introduced as soon as the fundamental theorems are available, and is immediately used to establish Taylor's Theorem. Power series are introduced early on as a class of functions generalizing polynomials. As continuity, integration, and differentiation are developed in turn, power series serve as a substantial family of examples to which new techniques may be applied. Continuity of functions is *defined* in terms of sequences, allowing the technical work on sequential limits to be used immediately. (As a fringe benefit, explicit mention of domain issues is deferred until the introduction of function limits.)

Generally, the goals at each stage are: to assist students in translating between the informal notions of calculus and the language of modern analysis; to reach interesting, substantial examples and theorems as quickly as possible; to build useful tools for subsequent application; and to "push the envelope" of what can be proven using available resources.

Major results are stated for ease of subsequent use rather than for maximum generality, and almost all are proven in the main body of the text. In rare cases that the proof of a "major result" (i.e., a theorem to be used subsequently) is relegated to the exercises, a parallel statement has been proven in the text in detail, so that students have a "working example" they can modify. This expository strategy is somewhat contrary to the conventional wisdom of the author's formative years, but smooths the learning experience while still requiring students to understand and absorb techniques of proof.

Despite the "storyline" structure of the book, certain topics may be omitted without loss of continuity. The procedural definition and properties of integer powers are familiar, and the proofs merely provide "realistic" examples of mathematical induction. The topology of sets is referenced mostly in the exercises; the notions of isolated points and limit points are used in the discussion of function limits, but may be defined specifically for that purpose. The material on cardinality is not used elsewhere. The Gamma function and sine integral are extended applications of a variety of techniques, but not used subsequently.

Modulo one's pacing and selective omission, the entire book can be covered comfortably in one 13-week term.

To the Student

Elementary analysis (i.e., calculus) studies functions of one real variable *at arbitrarily small scales.* The main idea is implicit in the concept of limits. In this book we define limits carefully enough to prove theorems, not merely to state intuitively plausible principles.

Real analysis requires understanding of *real numbers.* You have undoubtedly been familiar with the number line for years. However, the number line holds unsuspected complexity. A few examples will illustrate the subtleties of the real numbers and of limits. These should convince you of the *need* to study the real numbers and limits carefully. They should also serve as useful conceptual reference points throughout the course. Each definition we introduce is geared toward examining and analyzing similar classes of phenomena.

Is $0.9\overline{9} = 1$?

The expression $0.9\overline{9}$ stands for the real number whose decimal representation consists of an infinite string of 9s. Most people find it intuitively obvious that $0.9\overline{9} \leq 1$, but there is a surprisingly wide split of intuition on which alternative holds: $0.9\overline{9} < 1$ or $0.9\overline{9} = 1$. This question gets at the nature of fractions and infinite decimals, but also of limits. To emphasize one's typical state of ignorance after taking calculus, the discussion below uses quotation marks to surround familiar terms that require careful definition in this course.

The number $0.9\overline{9}$ is the "limit" of the "sequence" 0.9, 0.99, 0.999, 0.9999, …, in which "the pattern continues". Successive terms get "closer and closer" to 1, but each term differs from 1 by a definite, positive amount. "In the limit", is there a lingering positive difference, or is the difference $1 - 0.9\overline{9}$ exactly equal to zero?

This simple mathematical question has a well-defined answer. In other words, once we have carefully defined the concept of a "real number" and the concept of an "infinite decimal" (repeating or not), only one answer is logically consistent: The two *are* equal. One way to prove this is to write $x = 0.9\overline{9}$. Since multiplying by 10 shifts the decimal to the right one place, $10x = 9.9\overline{9}$. Now subtract the first from the second to deduce that $9x = 9$.

Interestingly, some who doubt that $0.9\overline{9} = 1$ find it intuitively clear that the differences $1 - 0.9 = 0.1$, $1 - 0.99 = 0.01$, $1 - 0.999 = 0.001$, ... do approach 0 (rather than approaching an infinitesimal positive limit). The intuitive difference may be that in the decimal expression $0.0\overline{0}$, the "correct" digits accumulate, while in $0.9\overline{9}$, "the last digit never carries".

Does $\sqrt{2}$ Exist?

By convention, the expression "$\sqrt{2}$" stands for a positive real number whose square is equal to 2. Nowadays, every child "knows", by asking their calculator, that "$\sqrt{2} = 1.414213562$". A moment's reflection, however, reveals the falsity of this belief: Multiplying $x = 1.414213562$ (or *any finite decimal* not representing an integer) by itself yields a finite decimal not representing an integer! For example, since the rightmost decimal of x is 2, the rightmost decimal of x^2 is 4, so x^2 is manifestly not exactly equal to 2.

In fact, the Ancient Pythagoreans knew that "$\sqrt{2}$ is irrational". More correctly, in modern terms, the Pythagoreans proved the "contrapositive": If $x = p/q$ is a rational number (ratio of integers) in lowest terms, then $x^2 \neq 2$. This inequality can be written as $p^2 \neq 2q^2$.

Each of the integers p and q is either even (an integer multiple of 2) or odd (leaves a remainder of 1 on division by 2). The square of an odd integer is odd, while the square of an even integer is an integer multiple of *four*. We can prove $p^2 \neq 2q^2$ by considering cases in which p and q are either even or odd.

p **odd.** If p is odd, then p^2 is odd. Since $2q^2$ is even regardless of q, $p^2 \neq 2q^2$ in this case.

p **even.** Since p/q is in lowest terms, q is odd in this case. But p^2 is an integer multiple of 4, while q^2 is odd, so $2q^2$ is not a multiple of 4. Again, $p^2 \neq 2q^2$.

In all cases, $p^2 \neq 2q^2$. We conclude that no rational number is a square root of 2.

Using unique factorization of an integer into primes, we can give an elegant argument that the only integers having a rational square root are already perfect squares (of integers). Let N be a positive integer. If $x = p/q$ in lowest terms and if $x^2 = N$, then rearranging as above gives $p^2 = Nq^2$. Now factor each side of this equation into primes. The prime factors of p^2 and q^2 appear with even exponent, so every prime factor of N appears with even exponent. This means N is a perfect square.

Though there is no rational number whose square is exactly 2, it is possible to construct "better and better rational approximations" to $\sqrt{2}$ using simple arithmetic operations. The recursive rule

$$x_0 = 2 = \frac{p_0}{q_0}, \quad x_{n+1} = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right) = \frac{1}{2}\left(\frac{p_n}{q_n} + \frac{2q_n}{p_n}\right) = \left(\frac{p_n^2 + 2q_n^2}{2p_nq_n}\right)$$

introduced in Example 6.71 generates successive terms

$$\begin{aligned} x_1 &= \frac{3}{2} = 1.5, & x_3 &= \frac{577}{408} \approx 1.41421568\ldots, \\ x_2 &= \frac{17}{12} = 1.416\bar{6}, & x_4 &= \frac{665857}{470832} \approx 1.4142135623\ldots. \end{aligned}$$

The numerators p_n and denominators q_n are growing "rapidly", and the sequence formally approaches ∞/∞. However, the decimal approximations seem to be "approaching a limit". If we call this limit x_∞, we expect that

$$x_\infty = x_{\infty+1} = \frac{1}{2}\left(x_\infty + \frac{2}{x_\infty}\right).$$

Algebra gives $x_\infty^2 = 2$, or $x_\infty = \sqrt{2}$ (since certainly $x_\infty \neq -\sqrt{2}$). This formal argument leaves important questions unanswered: Just what *is* a real number, and on what basis does the preceding sequence have a limit? How are we to work with specific real numbers if we are unable to construct an approximating sequence of rationals?

A more general argument will help motivate definitions to be made later. Consider the *set* A of positive real numbers x satisfying $x^2 < 2$. For example, $1 \in A$ (the real number 1 satisfies $1^2 < 2$) but $2 \notin A$ (since $2^2 \not< 2$). We will show that the set A is "bounded above": There exists a real number M such that $x \leq M$ for every positive real number x satisfying $x^2 < 2$. (For example, we may take $M = 2$ or $M = 1.5$.)

Due to a property of the real numbers, there exists a "least upper bound" of A, namely a real number ℓ that is (i) Greater than or equal to *every* number in A (i.e., is an upper bound of A) and (ii) Less than or equal to *every* upper bound of A. We then argue that $\ell^2 = 2$ exactly: If $\ell^2 < 2$ then property (i) is false, while if $\ell^2 > 2$ then property (ii) is false. The third alternative, $\ell^2 = 2$, is all that remains!

Continuous Functions

In calculus, you may have learned that a function f is "continuous" if the graph $y = f(x)$ can be drawn without lifting your pencil from the paper, namely if the graph has no "holes", "breaks", or "jumps". Unfortunately, a procedural criterion is unsuitable for proving theorems, and in any case this heuristic does not perfectly capture the formal analytic definition. We'll examine a few deficiencies in turn.

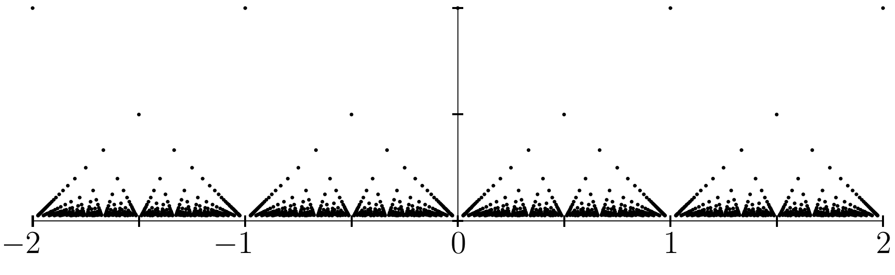
First, the graph of a "continuous function" *can* have holes, breaks, and/or jumps if the domain of the function is not an interval. The *signum* function $\operatorname{sgn}(x) = x/|x|$, defined for $x \neq 0$, is continuous at each point of its domain. "Near" every positive real number x, the function is equal to 1, and "near" every negative number x, the function is -1. There is no discontinuity at 0 because 0 is not in the domain.

Second, the graph of a continuous function may be impossible to draw at all because the "length" of the graph is infinite between *every pair of points*! This behavior is, in fact, *typical* for a continuous function.

Third, a function may be continuous at some points of its domain and discontinuous at others. As a fairly extreme example, consider

$$f(x) = \begin{cases} 1/q & \text{if } x = p/q \text{ in lowest terms,} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

The graph seems to consist entirely of "holes" and "jumps". However,



this function manages to be continuous at every irrational number. A proof is outlined in the exercises.

A “better approximation” to the formal definition of continuity is the following principle:

> If f is continuous at x, then $f(x')$ can be made as close as we like to $f(x)$ by taking x' sufficiently close to x.

Indeed, the formal definition “merely” makes precise what is meant by “as close as we like” and “sufficiently close”.

Properties of Continuous Functions

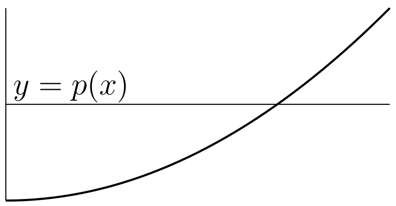
The preceding criterion makes sense in the rational number system. Nonetheless, the rational number system is unsuitable as a foundation for analysis. To see why, we’ll consider functions that are continuous at every point of an interval.

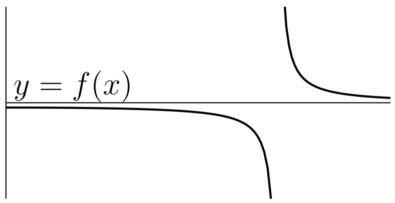
In calculus, you learned that if a continuous function is defined on a closed, bounded interval, and if the graph is below the x axis at one endpoint and above the x axis at the other endpoint, then the graph of f touches the x axis at least once, the *intermediate value theorem.*

You also learned that a continuous function defined on a closed, bounded interval achieves an absolute minimum value and an absolute maximum value, the so-called *extreme value theorem.*

These theorems are among the most useful yet subtle properties of continuous functions (of a *real* variable) that we will establish. Consider two functions of a *rational* variable:

$$p(x) = x^2 - 2 \quad \text{and} \quad f(x) = \frac{1}{x^2 - 2} \qquad \text{for } 0 \leq x \leq 2.$$





The function p is a polynomial, and therefore (as we will prove) is continuous in its domain, the closed *rational* interval $[0, 2]$. More remarkably, the function f is a quotient of non-zero (!) polynomials, and is therefore also continuous *everywhere* in the rational interval $[0, 2]$, despite its appearance of having a vertical asymptote.

The graph of the polynomial p starts below the x axis and ends above, since $p(0) = -2 < 0 < 2 = p(2)$. However, the graph of p does not touch the x axis, since $p(x) \neq 0$ for all rational x. *The intermediate value theorem fails in the rational universe.*

The extreme value theorem fails for f. Visually this is clear from the graph. In a bit more detail, let (x_n) be the rational sequence converging to $\sqrt{2}$ that was defined earlier. Since $x_n^2 - 2$ decreases to zero, $f(x_n)$ grows without bound. Consequently, f has no maximum. For similar reasons, f has no minimum.

Your inclination may be to dismiss these phenomena: “Well, then, don’t try to use the rational numbers as the setting for calculus”. The problem is, in what setting *are* we to develop calculus, and how are we to prove, e.g., the Intermediate Value Theorem in this setting?

Power Series

Among the simplest functions in calculus are polynomials, which may be evaluated using nothing more that the four arithmetic operations:

$$a_0 + a_1x + a_2x^2 + \dots + a_nx^n = a_0 + x\big[a_1 + x[a_2 + \dots x(a_n)]\big].$$

By definition, a polynomial has only finitely many summands. What if we relax this restriction?

In calculus, you met power series, including Taylor series, and encountered remarkable formulas such as

$$\begin{aligned}
\frac{1}{1-x} &= 1 + x + x^2 + x^3 + \dots + x^n + \dots && |x| < 1,\\
e^x &= 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^k}{k!} + \dots && x \text{ real},\\
\cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots + (-1)^k \frac{x^{2k}}{(2k)!} + \dots && x \text{ real},\\
\sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots + (-1)^k \frac{x^{2k+1}}{(2k+1)!} + \dots && x \text{ real}.
\end{aligned}$$

You learned that power series can be manipulated—added, subtracted, multiplied, differentiated, integrated, etc.—as if they were polynomials. You may even have seen seemingly-magical relationships, such as *Euler’s formula*

$$e^{i\theta} = \cos\theta + i\sin\theta,$$

a striking identity with observable physical consequences.

Close inspection reveals that *evaluating* a power series involves a limit (summing an infinite series), while *differentiating* or *integrating* entails a second limit (taking difference quotients, for example). The theorems guaranteeing that power series can be manipulated like polynomials boil down to the possibility of "exchanging" these limit operations: performing them in either order and obtaining the same result: $\lim\limits_{h\to 0}\lim\limits_{n\to\infty}$ versus $\lim\limits_{n\to\infty}\lim\limits_{h\to 0}$, for instance.

The Fundamental Theorems

A functional equation $y = f(x)$ may be interpreted as a relation of dependence of one quantity (the "dependent variable" y) on another (the "independent variable" x). Working purely formally, we can measure the "rate of change" of f by changing the input slightly and observing the resulting change in the output: $y + dy = f(x + dx)$. The "rate of change of y with respect to x" would be

$$\frac{\text{change in } y}{\text{change in } x} = \frac{dy}{dx} = \frac{f(x+dx) - f(x)}{dx}.$$

Conversely, suppose dy/dx is known, and we want to find the total change in y as x runs between two specified bounds, say a and b. Again formally, the "infinitesimal increment" of y is

$$dy = \frac{dy}{dx}\,dx = f(x+dx) - f(x).$$

Adding up these quantities, the sum formally telescopes, and we find

$$\begin{aligned}\text{Sum of } dy &= \text{Sum of } f(x+dx) - f(x) \text{ for } a \leq x \leq b \\ &= \big[f(a+dx) - f(a)\big] + \big[f(a+2\,dx) - f(a+dx)\big] + \dots \\ &\qquad\qquad + \big[f(b) - f(b-dx)\big] \\ &= f(b) - f(a) = \text{Total change in } y.\end{aligned}$$

Unfortunately, a moment's reflection reveals vexing snags: To get an exact value for the rate of change, we need dx to be infinitesimal, which for real numbers means "equal to 0". But when $dx = 0$, the quotient dy/dx is equal to $0/0$, and thus fails to be defined.

Similarly, the infinitesimal increments $dy = f(x + dx) - f(x)$ are either zero (so their sum is zero), or not (so they are merely approximate). Either way our formal calculations are suspect.

Analysis is the branch of mathematics that logically justifies our ability to have our cake (algebraically manipulate expressions that are formally infinitesimal) and eat it too (work only with real numbers). The fundamental idea is that of "arbitrarily close approximation", codified in the theory of limits.

Differential and *integral calculus* are the formal rules and algorithms that allow us to calculate with infinitesimals as if they were real numbers, ignoring the fact that no non-zero real number is infinitesimal.

Summary

Analysis can be difficult to absorb for many reasons. The familiar real number system must be examined carefully and put onto a firm logical, conceptual, and computational foundation. You must become fluent at manipulating inequalities, and must develop your intuition about intervals of numbers.

The heuristic definitions of limits and continuity you saw in calculus supply only the barest guidance, and must be fundamentally supplanted in order to prove theorems. Geometric intuition must not be discarded entirely, but neither can it be relied upon for more than suggestions for setting up and establishing algebraic and analytic statements.

Not only are the definitions in analysis genuinely complicated, but the gap between the definitions and the theorems is bridged with mathematical idioms you have not seen before. You must become familiar with proofs that assume an infinite number of hypotheses, and/or that deduce an infinite number of conclusions.

Finally, proofs of theorems in analysis typically involve more pieces, and more conceptual layers, than proofs in algebra. It's easy to get lost in the thicket of details even as you wander the familiar paths of differential and integral calculus.

Happily, you'll soon start to see patterns and recognize landmarks. With practice, you'll absorb the precise definition of a limit, and thereby attain a higher level of understanding of calculus. In time, you'll confidently apply these ideas in novel situations.

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Chapter 1

Logic and Sets

Mathematics encompasses a language for expressing quantitative and logical propositions, and computational tools for using known truths to deduce new ones. In order to read, understand, and write mathematics successfully, you must become fluent with the terminology, syntax, and idioms of this language. This chapter merely sketches the bare essentials needed for the remainder of the book.

1.1 Negation and Logical Connectives

A *statement* is a sentence having a *truth value*, T (True) or F (False). Contact with the external world can be made via experience, but in mathematics *true* and *false* may be viewed as undefined terms.

Example 1.1. -4 is an even integer.

The decimal expansion of π is non-repeating and contains the string '999999'. (True)

$2 + 2 = 5$. (False)

Example 1.2. Sentences that are *not* statements include "x is a positive real number" (whose truth value depends on x), and "10^{1000} is a large number" ("large" has not been given a precise meaning).

Conventionally, abstract statements are denoted P and Q.

Not. The *negation* of a statement P is its logical opposite $\neg P$. You may regard the negation as P preceded by the clause "It is not the case that…", but usually a more pleasant wording can be found.

Example 1.3. P: $2 + 2 = 4$. $\neg P$: $2 + 2 \neq 4$.

Let P and Q be statements. New statements can be constructed using the "logical connectives" *and*, *or*, and *implies*.

And. The statement "P and Q" has its ordinary meaning: The compound statement is true provided both P and Q are true, and is false otherwise.

Example 1.4. $2 + 2 = 4$ and $0 < 1$. (True)
$2 + 2 = 5$ and $0 < 1$. (False)
$2 + 2 = 5$ and $1 < 0$. (False)

Or. The statement "P or Q" always has the "inclusive" meaning in mathematics: P is true, or Q is true, *or both*.

Example 1.5. $2 + 2 = 4$ or $0 < 1$. (True)
$2 + 2 = 5$ or $0 < 1$. (True)
$2 + 2 = 5$ or $1 < 0$. (False)

Implies. A statement of the form "If P then Q", also read "P *implies* Q", is called a *logical implication* and plays a central role in mathematics. P is called the *hypothesis* of the implication, Q the *conclusion*.

By definition, a logical implication is *valid* provided Q is true whenever P is true. In other words, "P implies Q" is a valid deduction unless P is true and Q is false.

Example 1.6. If $1 \neq 0$, then $1^2 \neq 0$. (True)
If $1 \neq 0$, then $1^2 = 0$. (False)
If $1 = 0$, then $0 = 0$. (True)
If $1 = 0$, then $1^2 = 0$. (True)

If "P implies Q" is valid, we think of Q as being *deduced* or *derived* from P. The definition of "valid implication" ensures that by starting with true hypotheses and making valid deductions, we obtain only true conclusions, not falsehoods. There are two noteworthy and potentially confusing consequences of this convention, however.

First, it is valid (not logically erroneous) to deduce an arbitrary conclusion from a false hypothesis. An implication with false hypothesis is said to be *vacuously true*. Humorous examples abound: "If $1 = 0$, then money grows on trees."

In particular, the third and fourth implications of the preceding example are vacuously true. It may be helpful to point out that in each case, we can give a proof. If $1 = 0$, then subtracting this equation from itself gives $0 = 0$, which proves the third statement, while squaring gives $1^2 = 0^2 = 0$, proving the fourth statement.

Second, a valid implication need not connect causally related statements. The implication "If $0 = 0$, then 2 is an even integer" is valid because both the hypothesis and conclusion are true, but is effectively a *non sequitur*; the conclusion does not "follow" from the hypothesis in any obvious sense. A valid implication does not, of itself, constitute a proof. In the example at hand, we know the implication is valid only because there exists a separate proof, consisting of implications whose validity can be checked directly.

In these two senses, mathematicians are liberal in deeming an implication to be valid. Again, "validity" is the weakest criterion that excludes the act of drawing a false conclusion from a true hypothesis.

In this book, and throughout mathematics in practice, valid deductions do actually link causally related statements. Most implications involve classes of objects, and assert that every object satisfying some condition must also satisfy some other condition.

Negation and Conjunctions

If P and Q are statements, then the statement "P and Q" is false if *at least one* of P and Q is false. If someone assures you two statements are both true, only one has to be false for the assurance to be unfounded. Formally, the compound statements

$$\neg(P \text{ and } Q), \qquad (\neg P) \text{ or } (\neg Q)$$

express the same logical condition.

Analogously, if someone assures you at least one statement of two is true, then both must be false for the assurance to be unfounded. Formally, the compound statements

$$\neg(P \text{ or } Q), \qquad (\neg P) \text{ and } (\neg Q)$$

express the same logical condition.

Together, the two preceding relationships are known as *De Morgan's laws*, after the 19th Century English logician A. De Morgan. Loosely, the conjunctions "and" and "or" are interchanged by negation, perhaps contrary to first impression.

Consequently, the order of negation and a connective matters:

Example 1.7. The integers 1 and 0 are **not both** zero. (True.)

The integers 1 and 0 are **both not** zero. (False.)

1.2 Quantification

To accommodate classes of objects in the framework of statements, we allow statements to contain *variables* standing for elements of a set, so long as each variable is "quantified", accompanied by the phrase "for every" or "there exists". The quantifiers are crucial; pay close attention to them while reading, and *do not omit them when thinking and writing.*

Example 1.8. For every real number x, $0 < 1 + x^2$. (True)
For every real number x, $0 < x^2$. (False)
For every integer n, $n^2 = 1$. (False)

The preceding "for every" statements involve *universal quantification.* Each statement encapsulates multiple statements. For example, the first statement of the preceding example encapsulates an infinite collection of statements, one for each real number.

Example 1.9. There exists a real number x such that $0 < 1 + x^2$. (True)
There exists a real number x such that $0 < x^2$. (True)
There exists an integer n such that $n^2 = 1$. (True)
There exists a real number x such that $x = x + 1$. (False)

The preceding "there exists" statements involve *existential quantification.* Again, each encapsulates multiple statements. For example, the fourth expresses that at least one truism is found among the statements. The compound statement is false because *every* individual statement is false.

Remark 1.10. The statements of the preceding examples contain only "bound" (i.e., quantified) variables.

Sentences containing "free" or "unbound" variables (such as "n is an even integer" or "$x^2 + x - 2 = 0$") are not statements. However, sentences containing unbound variables play the useful role of *conditions* in mathematics, selecting objects (perhaps integers n or real numbers x) for which the resulting statement is true.

Many mathematical theorems take the universally quantified form "For every x satisfying $P(x)$, condition $Q(x)$ is true". For stylistic variety, such statements may be worded as implications involving "arbitrary" values of variables.

Example 1.11. If x is a real number such that $x^2 + x - 2 = 0$, then $x = 1$ or $x = -2$. (True)

If n is an integer, then there exist unique integers q and r such that $n = 4q + r$ and $0 \leq r < 4$. (True)

If a, b, and c are positive integers, then $a^3 + b^3 \neq c^3$. (True)

Quantifiers and Negation

The universal quantifier "for every" may be viewed as an enhancement of the "and" conjunction: "For every integer n, the condition $P(n)$ is true" means that the infinitely many statements $P(0)$, $P(1)$, $P(-1)$, and so forth, are *all* true.

The existential quantifier "there exists" may be viewed similarly as an enhancement of "or": "There exists an integer n such that the condition $P(n)$ is true" means that among the infinitely many statements $P(0)$, $P(1)$, $P(-1)$, …, *at least one* is true.

Example 1.12. Logical negation "converts" a "for every" statement into a "there exists" statement of negations, and converts a "there exists" statement into a "for every" statement of negations:

P: For every real number x, $x^2 \geq 0$.
$\neg P$: There exists a real number x such that $x^2 < 0$.

P: There exist real numbers x and y such that $x^2 + y^2 = 1$.
$\neg P$: For all real numbers x and y, $x^2 + y^2 \neq 1$.

Remark 1.13. This type of linguistic transformation needs to become second nature. Particularly, a positive assertion regarding a class of objects can be disproved by finding a counterexample, but cannot be proved by finding an example.

Remark 1.14. When the hypothesis of a logical implication contains a variable but no quantifier is explicitly present, the convention is to read "for every". For example, "If $x > 0$ then $x^2 > 0$" should be read "For every real number x, if $x > 0$ then $x^2 > 0$" (assuming context dictates x is a real number).

If an implicitly-quantified statement is negated, the existential quantifier must be added explicitly: "There exists a real number $x > 0$ such that $x^2 \leq 0$".

To avoid confusion, including your own, include logical quantifiers explicitly. This book makes a special effort to set a good example.

Implications, and Multiple Quantifiers

Among the most subtle conditions in mathematics are those containing multiple quantifiers. Pay careful attention to the *ordering* of quantifiers in a single sentence. Changing the order of words and phrases can completely change the meaning of a sentence. The fundamental definitions of analysis, such as Definition 4.7, are conditions whose meaning depends delicately on the precise wording. When you encounter multiply-quantified statements, slow down and read several times to ensure you thoroughly understand the dependencies implicit in the ordering.

Example 1.15. For every real number x, there exists an integer N such that $x < N$. (True; every real number x is smaller than some integer N.)

There exists an integer N such that for every real x, $x < N$. (False; there is no single integer N greater than every real number x.)

1.3 Truth Tables and Applications

The logical operators ("not", "and", "or", and "implies") introduced above are neatly summarized by *truth tables*:

P	Q	$\neg P$	P and Q	P or Q	P implies Q
T	T	F	T	T	T
T	F	F	F	T	F
F	T	T	F	T	T
F	F	T	F	F	T

Truth tables furnish a useful tool for studying sentences built of other statements and logical connectives. This section gives a few applications.

Logical Equivalence. Two statements P and Q are *logically equivalent* if each implies the other: P implies Q and Q implies P. For brevity, we may write P iff Q, "iff" being short for "if and only if". A truth table calculation shows P and Q are equivalent precisely when they have the same truth value:

P	Q	P implies Q	Q implies P	P iff Q
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

The Converse. The implications "P implies Q" and "Q implies P" are said to be *converse* to each other. The preceding table shows these implications are not equivalent.

The Contrapositive. The implications "P implies Q" and "$\neg Q$ implies $\neg P$" are said to be *contrapositive* to each other. An implication and its contrapositive are logically equivalent:

P	Q	P implies Q	$\neg Q$	$\neg P$	$\neg Q$ implies $\neg P$
T	T	T	F	F	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

This fact of logic should become second nature to you. Many implications are easier to understand and prove in contrapositive form.

Example 1.16. In each statement, x stands for a real number. Let P be the statement "$x^2 - 1 \neq 0$" and Q be the statement "$x \neq 1$".

The implication P implies Q is true, but may require a few seconds' thought to see.

The converse implication, "If $x \neq 1$, then $x^2 - 1 \neq 0$" is an invalid deduction. The number $x = -1$ is a *counterexample*: It satisfies the converse hypothesis Q, but not the converse conclusion P.

The contrapositive reads, "If $x = 1$, then $x^2 - 1 = 0$." This implication is obviously true, and on general grounds its truth implies the truth of P implies Q.

Example 1.17. In each statement, $x \geq 0$ is a real number, and n is a positive integer.

Direct implication: If $x < 1/n$ for every n, then $x = 0$.

Converse: If $x = 0$, then $x < 1/n$ for every n.

Contrapositive: If $x > 0$, then there exists an n such that $1/n \leq x$.

It turns out that all three statements are true. The second is easily seen, even though the conclusion consists of infinitely many statements: $0 < 1$, $0 < 1/2$, $0 < 1/3$, etc.

The third statement is true, and not difficult to see; informally, $1/k \to 0$ as $k \to \infty$, so if $x > 0$, there is some positive integer n such that $1/n \leq x$.

The direct implication is therefore true, since its contrapositive is true. However, the direct implication exhibits a new phenomenon: The hypothesis consists of infinitely many statements, $x < 1$, $x < 1/2$,

$x < 1/3$, etc., but *no finite number of these statements implies the conclusion*. Indeed, if we assume only finitely many inequalities of the form $x < 1/n$, there is a largest denominator, say N, and our collection of inequalities is equivalent to the single inequality $x < 1/N$, which does not imply $x = 0$.

1.4 Sets

Modern mathematics is built on the concept of a "set", a collection of "elements". These primitive notions will serve in lieu of definitions.

Abstract sets will be denoted with capital letters, such as A or B. Elements are normally denoted with lower case letters, such as a and b. We write $a \in A$ as shorthand for "a is an element of (the set) A", and $b \notin A$ for the logical negation "b is not an element of A".

Definition 1.18. Let A and B be sets. We say A is a *subset* of B, written $A \subseteq B$, if $x \in A$ implies $x \in B$, that is, if every element of A is an element of B.

Two sets A and B are *equal* if $A \subseteq B$ and $B \subseteq A$, namely if they have exactly the same elements: $x \in A$ if and only if $x \in B$.

Example 1.19. Let A be a set. For each element a in A, there is a *singleton* set $\{a\}$ contained in A. Take care to distinguish a and $\{a\}$; a is an object, while $\{a\}$ is a "package" that contains exactly one object.

Example 1.20. There exists an *empty set* $\varnothing$ containing *no* elements. For all x, the clause $x \in \varnothing$ is false. In particular, for every set A the logical implication "$x \in \varnothing$ implies $x \in A$" is vacuous (has false hypothesis). Consequently, $\varnothing \subseteq A$ is true for all A.

Example 1.21. The symbol $\mathbf{R}$ denotes the set of *real numbers*, to be defined precisely in Chapter 2. The symbol $\mathbf{Z}$ denotes the set of *integers*, or *whole numbers*.

In mathematics, we often want to restrict our attention to sets that are contained in a fixed set $\mathcal{U}$, sometimes called a *universe*. Specific subsets of $\mathcal{U}$ are conveniently described using *set-builder notation*, in which elements are selected according to logical conditions formally known as a *predicates*. The expression $\{x \text{ in } \mathcal{U} : P(x)\}$ is read "the set of all x in $\mathcal{U}$ such that $P(x)$".

To personify, if $\mathcal{U}$ is a population whose elements are individuals, then a subset A of $\mathcal{U}$ is a club or organization, and the predicate defining A is a membership card. We screen individuals x for membership in A by checking whether or not x carries the membership card for A, namely whether or not $P(x)$ is true.

Example 1.22. The expression $\{x \text{ in } \mathbf{Z} : x = 2n \text{ for some } n \text{ in } \mathbf{Z}\}$ is the set of *even integers*.

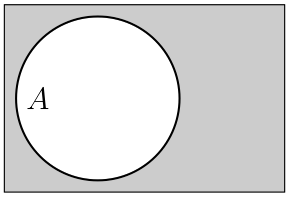
Remark 1.23. For brevity, we sometimes write, e.g., the set of even integers as $\{2n : n \in \mathbf{Z}\}$, read "the set of $2n$ such that n is an integer". This way of writing a set is convenient, and the meaning is generally clear, but it isn't technically proper grammar. To define a set formally, first give the universe, then specify the predicate.

Sets and Logic

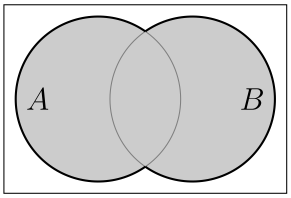
Let $\mathcal{U}$ be a universe, and let A and B be subsets of $\mathcal{U}$. The clauses $x \in A$ and $x \in B$ may be viewed as predicates P and Q on elements of $\mathcal{U}$. By definition, the logical implication "$x \in A$ implies $x \in B$" corresponds to the set relation $A \subseteq B$. Logical negation, disjunction (or), and conjunction (and) similarly have natural interpretations in terms of A and B.

The *complement* of A:	$A^c = \{x \text{ in } \mathcal{U} : x \notin A\}$.
The *union* of A and B:	$A \cup B = \{x \text{ in } \mathcal{U} : x \in A \text{ or } x \in B\}$.
The *intersection* of A and B:	$A \cap B = \{x \text{ in } \mathcal{U} : x \in A \text{ and } x \in B\}$.

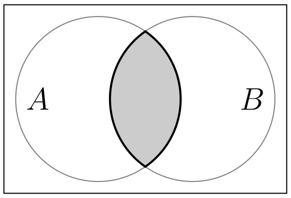
A *Venn diagram* represents subsets of a universe $\mathcal{U}$ pictorially. The universe is depicted as a rectangle, and subsets are disks or, if necessary, more complicated shapes. The complement of A, or the union and intersection of two sets A and B, might be drawn as indicated:



A^c



$A \cup B$



$A \cap B$

Definition 1.24. Let A and B be subsets of $\mathcal{U}$. Their *difference* is defined to be $A \setminus B = \{x \text{ in } A : x \notin B\} = A \cap B^c$.

We say A and B are *disjoint* if $A \cap B = \varnothing$, namely if A and B have no elements in common.

The *power set* of A, $\mathscr{P}(A)$, is the set of all subsets of A.

Example 1.25. If $A = \{0, 1\}$ has two elements, the power set $\mathscr{P}(A)$ has four elements: $\mathscr{P}(A) = \big\{\varnothing, \{0\}, \{1\}, A\big\}$. The empty set and A itself are always subsets of A, a.k.a., *elements* of $\mathscr{P}(A)$, so a power set is never empty. For example, $\mathscr{P}(\varnothing) = \{\varnothing\}$ has a single element.

Families of Sets

Let I be a non-empty set (possibly a set containing infinitely many elements), and suppose we have a family of sets A_i, one for each i in I, with each A_i contained in some fixed universe $\mathcal{U}$. We denote the family of sets $\{A_i\}_{i\in I}$, and say the family is *indexed by* I.

Definition 1.26. Let $\{A_i\}_{i\in I}$ be a family of subsets of $\mathcal{U}$. We define the *union* and *intersection* of the family to be the following subsets of $\mathcal{U}$:

$$\bigcup_{i\in I} A_i = \{x \text{ in } \mathcal{U} : x \in A_I \text{ for } \textit{some } i \text{ in } I\},$$
$$\bigcap_{i\in I} A_i = \{x \text{ in } \mathcal{U} : x \in A_I \text{ for } \textit{all } i \text{ in } I\}.$$

Remark 1.27. If $I = \{1, 2, 3, \dots, n\}$, a family of sets indexed by I is often denoted $\{A_i\}_{i=1}^{n}$, and the union and intersection are denoted

$$\bigcup_{i\in I} A_i = \bigcup_{i=1}^{n} A_i, \qquad \bigcap_{i\in I} A_i = \bigcap_{i=1}^{n} A_i.$$

Definition 1.28. Let $\mathbf{N}$ be the set of non-negative integers, and let $\{A_k\}_{k\in\mathbf{N}}$ be a family of sets. We say $\{A_k\}_{k\in\mathbf{N}}$ is *nested outward* if $A_k \subseteq A_{k+1}$ for every k in $\mathbf{N}$.

We say $\{A_k\}_{k\in\mathbf{N}}$ is *nested inward* if $A_{k+1} \subseteq A_k$ for every k in $\mathbf{N}$.

Exercises

Exercise 1.1. Let A, B, and C be subsets of a universe $\mathcal{U}$, and let P, Q, and R be the clauses $x \in A$, $x \in B$, and $x \in C$. Use truth tables to establish the indicated identities.

(a) $(A \cup B) \cup C = A \cup (B \cup C)$.

(b) $(A \cap B) \cap C = A \cap (B \cap C)$.

Exercise 1.2. Let A and B be subsets of $\mathcal{U}$.

(a) Prove $A \subseteq B$ if and only if $B^c \subseteq A^c$, and illustrate with a Venn diagram.

(b) How is part (a) related to contrapositives?

Exercise 1.3. Let P and Q be arbitrary statements. Use a truth table to prove "P implies Q" is logically equivalent to "$\neg P$ or Q".

Exercise 1.4. Let P, Q, and R be arbitrary statements. Use a truth table to prove the following pairs of statements are logically equivalent:

(a) "$\neg(P$ or $Q)$" and "$\neg P$ and $\neg Q$".

(b) "$\neg(P$ and $Q)$" and "$\neg P$ or $\neg Q$".

(c) "$(P$ or $Q)$ and R" and "$(P$ and $R)$ or $(Q$ and $R)$".

(d) "$(P$ and $Q)$ or R" and "$(P$ or $R)$ and $(Q$ or $R)$".

Exercise 1.5. Let A, B and C be subsets of a universe $\mathcal{U}$. As in Exercise 1.1, use truth tables to establish *De Morgan's laws* (a) and (b) and the *distributive laws* (c) and (d). How is each part related to the corresponding part of the preceding exercise?

(a) $(A \cup B)^c = A^c \cap B^c$.

(b) $(A \cap B)^c = A^c \cup B^c$.

(c) $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$.

(d) $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$.

Exercise 1.6. For each positive real number r, let

$$A_r = (-r, r) = \{x \text{ in } \mathbf{R} : -r < x < r\}.$$

"Simplify" the following sets, i.e., write each without using a union or intersection sign. It will be necessary to argue intuitively.

(a) $\displaystyle\bigcup_{r>0} A_r$. (b) $\displaystyle\bigcap_{r>0} A_r$.

Exercise 1.7. For each positive integer n, let

$$A_n = (-n, n) = \{x \text{ in } \mathbf{R} : -n < x < n\}.$$

Simplify the following sets.

(a) $\bigcup_{n>0} A_n$. (b) $\bigcap_{n>0} A_n$.

Exercise 1.8. For each positive integer n, let

$$A_n = (-\tfrac{1}{n}, \tfrac{1}{n}) = \{x \text{ in } \mathbf{R} : -\tfrac{1}{n} < x < \tfrac{1}{n}\}.$$

Simplify the following sets.

(a) $\bigcup_{n>0} A_n$. (b) $\bigcap_{n>0} A_n$.

Exercise 1.9. Let $\{A_k\}_{k\in\mathbf{N}}$ be a family of sets that is nested outward. If $n \in \mathbf{N}$, simplify the sets, and illustrate with a Venn diagram.

(a) $\bigcup_{k=0}^{n} A_k$. (b) $\bigcap_{k=0}^{n} A_k$.

Exercise 1.10. Let $\{A_k\}_{k\in\mathbf{N}}$ be a family of sets. For each n in $\mathbf{N}$, define

$$B_n = \bigcup_{k=0}^{n} A_k, \qquad C_n = \bigcap_{k=0}^{n} A_k.$$

(a) Show that the family $\{B_n\}_{n\in\mathbf{N}}$ is nested outward.

(b) Show that the family $\{C_n\}_{n\in\mathbf{N}}$ is nested inward.

Exercise 1.11. Let $\{A_i\}_{i\in I}$ be a family of subsets of some set $\mathcal{U}$, and assume $B \subseteq \mathcal{U}$. State and prove De Morgan's laws and the distributive laws in this setting. (See Exercise 1.5.)

Chapter 2

Numbers

In school you learned about "real numbers" and the "number line". While geometric intuition is an invaluable guide, it's not adequate as a logical foundation for limits, continuity, and calculus with the degree of logical care we need.

The bare legal contract for the real number system is laid out below as a list of axioms. The remainder of the chapter is devoted to (mostly familiar) consequences of the algebraic and order axioms. Completeness is discussed in Chapter 3.

The axioms for the real number system fall into three categories:

Algebraic Properties. (A1.–A4., M1.–M4., D.) The algebraic axioms concern the operations of addition and multiplication: What properties each operation has (associativity, commutativity, existence of an identity element and inverses), and how the two operations interact (the distributive law).

Order Properties. (O1.–O3.) These three axioms formalize the idea of one real number being "greater than" or "less than" another, and the fact that every pair of real numbers is "comparable". Geometrically, the order axioms guarantee that the real number system can be visualized as a subset of a line. "Less than" means "lies to the left of", and "greater than" means "lies to the right of".

Completeness. This axiom formalizes the geometric intuition that the real number system "has no gaps", or that "any quantity that can be approximated by real numbers is itself a real number". Geometrically, if A is a non-empty set on the number line, and if *some* point M lies to the right of every point of A, then there exists a *leftmost* point (i.e., smallest number) lying on or to the right of every element of A.

2.1 Axioms for the Real Numbers

Definition 2.1. The *real number system* consists of a non-empty set $\mathbf{R}$, two binary operations, $+$ and $\cdot$, and a subset P of $\mathbf{R}$ (the set of "positive" real numbers) satisfying the following thirteen axioms:

A1. Addition is associative: For all x, y, z in $\mathbf{R}$, $x+(y+z) = (x+y)+z$.

A2. Additive identity: There exists a unique element 0 in $\mathbf{R}$ such that for all x in $\mathbf{R}$, $0 + x = x + 0 = x$.

A3. Additive inverses: For every x in $\mathbf{R}$, there exists a unique $-x$ in $\mathbf{R}$ such that $x + (-x) = (-x) + x = 0$.

A4. Addition is commutative: For all x, y, in $\mathbf{R}$, $x + y = y + x$.

M1. Multiplication is associative: For all x, y, z in $\mathbf{R}$, $x\cdot(y\cdot z) = (x\cdot y)\cdot z$.

M2. Multiplicative identity: There exists a unique element 1 in $\mathbf{R}$, distinct from 0, such that for all x in $\mathbf{R}$, $1 \cdot x = x \cdot 1 = x$.

M3. Multiplicative inverses: For every non-zero x in $\mathbf{R}$, there exists a unique x^{-1} in $\mathbf{R}$ such that $x \cdot x^{-1} = x^{-1} \cdot x = 1$.

M4. Multiplication is commutative: For all x, y, in $\mathbf{R}$, $x \cdot y = y \cdot x$.

D. Multiplication (on the left) distributes over addition: For all x, y, z in $\mathbf{R}$, $x \cdot (y + z) = (x \cdot y) + (x \cdot z)$.

O1. The law of trichotomy: For every real number x, *exactly one* of the following holds: $x \in P$, $-x \in P$, or $x = 0$.

O2. Closure under addition: If x and y are in P, then $x + y$ is in P.

O3. Closure under multiplication: If x and y are in P, then $x \cdot y$ is in P.

C. Completeness: If A is a non-empty subset of $\mathbf{R}$ that is bounded above, then A has a least upper bound in $\mathbf{R}$.

Remark 2.2. The multiplication dot is often omitted: $x \cdot y = xy$.

Remark 2.3. These axioms have minor redundancies built in for convenience. For example, if $x+y = 0$, then by the commutative Axiom A4., $y + x = 0$ as well; there is no need to assume both equations in A3. Further, the uniqueness conditions in the axioms for identity elements and inverses can be deduced from the other axioms, see the proof of Lemma 2.14.

In fact, the set of real numbers, including the operations of addition and multiplication and the set of positive numbers, can be constructed from a much smaller number of axioms. Our axioms above would then be proven as theorems.

Auxiliary Concepts

We define the operations of *subtraction* and *division* in terms of addition and multiplication.

Definition 2.4. If x and y are real numbers, we define their *difference* to be $x - y = x + (-y)$.

If $y \neq 0$, we define their *quotient* to be $x/y = x \cdot y^{-1}$. In particular, $1/y = y^{-1}$.

Remark 2.5. Subtraction and division are neither associative nor commutative, as you should check.

We define the concepts of *positive* and *negative* numbers, and the relations *greater-than* and *less-than*, using Axioms O1.–O3.

Definition 2.6. Let x, y and z be real numbers.

If $z \in P$, we say z is *positive*, or 0 is less than z, and write $0 < z$.

If $y - x \in P$, we say x is *less than* y and write $x < y$. If $x < y$ or $x = y$, we say x is *less than or equal to* y, and write $x \leq y$.

Remark 2.7. If $0 < z$, we also say z is *greater than* 0 and write $z > 0$.

If $x < y$, we also say y is *greater than* x and write $y > x$. If $x \leq y$ we also say y is *greater than or equal to* x and write $y \geq x$.

Whole numbers and fractions constitute some of the most important classes of real numbers.

Definition 2.8. The set $\mathbf{N}$ of *natural numbers* is the smallest subset of $\mathbf{R}$ satisfying the following conditions: (i) $0 \in \mathbf{N}$; (ii) If $x \in \mathbf{N}$, then $x + 1 \in \mathbf{N}$.

A real number x is an *integer* or *whole number* if either x or $-x$ is a natural number. The set of integers is denoted $\mathbf{Z}$, from the German *Zahl*.

A real number x is a *rational number* if there exist integers p and q such that $q > 0$ and $x = p/q$. The set of rational numbers (a.k.a. *ratios* or *quotients*) is denoted $\mathbf{Q}$.

A real number x is *irrational* if x is not a rational number.

Remark 2.9. In this book, 0 is a natural number. The set of positive integers is denoted $\mathbf{Z}^+$.

Remark 2.10. Though numbers encountered in practice are often rational, "most" real numbers are irrational in a precise technical sense, see Example 5.45, *ff.* We have already encountered $\sqrt{2}$ (p. vi). Other examples include e (the base of the natural logarithms) and 2π (the period of the circular functions, a.k.a. the circumference of a unit circle).

Remark 2.11. In mathematics, a *field* is a set F together with two arithmetic operations $+$ and $\cdot$ that satisfy Axioms A1.–A4., M1.–M4., and D. An *ordered field* additionally satisfies Axioms O1.–O3.

The field of real numbers is the starring example in analysis. The set of rational numbers is an ordered field as well (see below), and plays a strong supporting role.

The set of integers is not a field; Axiom M3 fails.

The set of complex numbers is a field, but not an ordered field. By Corollary 2.19 (ii) below, in an ordered field, if $x \neq 0$, then $x^2 \in P$. However, each of 1 and -1 is the square of some complex number, so the trichotomy axiom does not hold.

2.2 Algebraic Properties

Though the axioms for the real numbers are numerous, we must still establish properties you may find "obvious". Doing so gives us a chance to see the axioms at work.

We first collect a few useful consequences of Axioms A1.–D. These properties hold not only in the real number system, but in any setting where Axioms A1.–D. are true, such as in the rational number system.

Lemma 2.12. *Multiplication on the right distributes over addition: For all x, y, z in $\mathbf{R}$, $(y + z) \cdot x = (y \cdot x) + (z \cdot x)$.*

Proof. If x, y, and z are arbitrary real numbers, then

$$\begin{aligned} (y + z) \cdot x &= x \cdot (y + z) && \text{Axiom M4.,} \\ &= (x \cdot y) + (x \cdot z) && \text{Axiom D.,} \\ &= (y \cdot x) + (z \cdot x) && \text{Axiom M4.} \end{aligned}$$ □

Remark 2.13. Because of the commutativity axioms, any identity involving addition or multiplication on the left has a corresponding version with the operation on the right.

Lemma 2.14. *Let x, y, and z be real numbers.*

(i) *If $x+y=x+z$, then $y=z$.*

(ii) *If $x\neq 0$ and $xy=xz$, then $y=z$.*

Proof. (i) Suppose $x+y=x+z$, and let $-x$ be the additive inverse of x. We have

$$\begin{aligned}
y &= 0+y && \text{Axiom A2.,}\\
&= \big((-x)+x\big)+y && \text{Axiom A3.,}\\
&= (-x)+(x+y) && \text{Axiom A1.,}\\
&= (-x)+(x+z) && \text{Hypothesis,}\\
&= \big((-x)+x\big)+z && \text{Axiom A1.,}\\
&= 0+z && \text{Axiom A3.,}\\
&= z && \text{Axiom A2.}
\end{aligned}$$

(ii) Mimic the proof of (i), replacing addition by multiplication and replacing the negative $-x$ by the reciprocal x^{-1}. □

Remark 2.15. These conclusions are called the "left cancellation laws". There are corresponding "right cancellations laws". For practice, state and prove these identities for yourself.

Lemma 2.16. *If $x\in\mathbf{R}$, then $-(-x)=x$. If $x\neq 0$, then $(x^{-1})^{-1}=x$, i.e., $1/(1/x)=x$.*

Proof. The equation $x+(-x)=0$ may be interpreted as saying that the (unique) additive inverse of $-x$ is x itself, i.e., $-(-x)=x$. Similarly, if $x\neq 0$, the equation $x(x^{-1})=1$ says $(x^{-1})^{-1}=x$. □

Proposition 2.17. *For all real numbers x and y:*

(i) $0\cdot x=x\cdot 0=0$.

(ii) *If $xy=0$, then $x=0$ or $y=0$.*

(iii) $-x=(-1)\cdot x=x\cdot(-1)$.

(iv) $(-x)(-y)=xy$. *Particularly,* $(-1)(-1)=1$.

Proof. (i) Taking $x = 0$ in A2, we have $0 = 0 + 0$. Now let x denote an arbitrary real number. Multiplying on the right by x and using the distributive law, we have

$$0 + (0 \cdot x) = (0 \cdot x) = (0 + 0) \cdot x = (0 \cdot x) + (0 \cdot x).$$

By the cancellation law, $0 = 0 \cdot x$. By the commutative Axiom M4., $0 = x \cdot 0$ as well.

(ii) Suppose $xy = 0$. If $x = 0$, there is nothing to prove. If $x \neq 0$,

$$y = (x^{-1}x)y = x^{-1}(xy) = x^{-1}(0) = 0.$$

(iii) Multiply $0 = \bigl(1 + (-1)\bigr)$ on the right by x:

$$x + (-x) = 0 = 0 \cdot x = \bigl(1 + (-1)\bigr) \cdot x = 1 \cdot x + (-1) \cdot x = x + (-1) \cdot x.$$

By cancellation, $-x = (-1) \cdot x$. By commutativity, $-x = x \cdot (-1)$.

(iv) Taking $x = -1$ in (iii) and invoking Lemma 2.16, we have $(-1)(-1) = -(-1) = 1$. Thus

$$(-x)(-y) = \bigl((-1)x\bigr)\bigl((-1)y\bigr) = (-1)(-1)(xy) = xy. \qquad \square$$

2.3 Order Properties

Mathematical inequalities are the bread and butter of analysis. The following rules for manipulating inequalities will be used continually. For example, (i) says that inequality is "transitive": inequalities can be "daisy-chained"; (iii) says that multiplying an inequality by a positive number preserves the inequality.

Theorem 2.18. *For all real numbers x, y, and z:*

(i) *If $x < y$ and $y < z$, then $x < z$.*

(ii) *If $x < y$, then $x + z < y + z$.*

(iii) *If $x < y$ and $0 < z$, then $xz < yz$.*

(iv) *If $x < y$ and $z < 0$, then $yz < xz$.*

(v) *If $0 < x < y$, then $0 < 1/y < 1/x$.*

Two additional extensions are useful occasionally:

Corollary 2.19. *Let x, y, z, and w be real numbers.*

(i) *If $x < y$ and $w < z$, then $x + w < y + z$.*

(ii) *If $0 < x < y$ and $0 < w < z$, then $0 < xw < yz$.*

In particular, $0 \leq x \cdot x$, with equality if and only if $x = 0$.

Remark 2.20. In words, (i) says two inequalities can be "added", preserving the sense of the inequality; (ii) expresses a similar guarantee when inequalities are multiplied, provided all quantities involved are positive.

Remark 2.21. The set P of numbers mentioned by Axioms O1.–O3. is uniquely characterized by these axioms. Specifically, P turns out to be precisely the set

$$\{y \text{ in } \mathbf{R} \mid y = x^2 \text{ for some } x \neq 0\}$$

of squares of non-zero real numbers. Corollary 2.19 (ii) guarantees every non-zero square is positive. The reverse implication, that every positive real number is the square of some real number, will be proven in Chapter 6.

To reduce the properties in Theorem 2.18 to Axioms O1.–O3., it is convenient first to establish special cases where one comparand is zero:

Lemma 2.22. *For all real numbers x and y,*

(i) *If $0 < x$ and $y < 0$, then $xy < 0$.*

(ii) *If $x < 0$ and $y < 0$, then $0 < xy$.*

(iii) *If $0 < x$, then $0 < 1/x$.*

Proof. (i) By trichotomy, $y < 0$ if and only if $-y \in P$. By Axiom O3. and Proposition 2.17 (iii), $-(xy) = x(-y) \in P$, i.e., $xy < 0$.

(ii) If $-x$ and $-y \in P$, then by Axiom O3. and Proposition 2.17 (iv), $0 < (-x)(-y) = xy$.

(iii) By trichotomy, either $1 \in P$ or $-1 \in P$. Since P is closed under multiplication, either $1 \cdot 1 = 1$ is positive, or $(-1) \cdot (-1) = 1$ is positive. Under either alternative, we conclude 1 is positive, or $0 < 1$.

Suppose $0 < x$. If x^{-1} were negative, then by (i), we would have $1 = x(x^{-1}) < 0$, which we have just seen is false. Contrapositively, if $0 < x$, then $0 < x^{-1}$. □

Proof of Theorem 2.18. (i) By definition, $x < y$ if and only if $y-x \in P$, and $y < z$ if and only if $z - y \in P$. By Axiom O2.,

$$z - x = (z - y) + (y - x) \in P,$$

which is equivalent to $x < z$.

(ii) For all x, y, and z, we have $y - x = (y + z) - (x + z)$. The claim follows immediately.

(iii) If $x < y$ and $0 < z$, then $y-x \in P$ and $z \in P$, so by Axiom O3., their product $(y - x)z = yz - xz$ is in P, which means $xz < yz$.

(iv) Since $x < y$ and $0 < -z$, part (iii) gives

$$0 < (y - x)(-z) = (x - y)z = xz - yz,$$

i.e., $yz < xz$.

(v) By Lemma 2.22 (iii), if $0 < x < y$, then $0 < 1/x$ and $0 < 1/y$. Algebra gives $1/x - 1/y = (y - x)/(xy) > 0$, i.e., $1/y < 1/x$. □

Proof of Corollary 2.19. (i) By Theorem 2.18 (ii), adding x to $w < z$ and adding z to $x < y$ gives $x + w < x + z < y + z$.

(ii) follows similarly from Theorem 2.18 (iii).

For the assertion about real squares, it suffices to note that $0 \cdot 0 = 0$ by Proposition 2.17 (i), while we have just shown that if $x \neq 0$, then $0 < x \cdot x$. □

2.4 Exponentiation

Definition 2.23. Let x be a non-zero real number. We define *integer powers* of x by

$$x^0 = 1, \qquad \left.\begin{aligned} x^{k+1} &= x^k \cdot x, \\ x^{-k} &= (x^k)^{-1}, \end{aligned}\right\} \quad \text{if } k \text{ is a natural number.}$$

Example 2.24. Expanding the definition for $x = 2$, we have

$$\begin{aligned} 2^3 &= 2^2 \cdot 2 = 2^1 \cdot 2 \cdot 2 = 2^0 \cdot 2 \cdot 2 \cdot 2 = 1 \cdot 2 \cdot 2 \cdot 2 = 2 \cdot 2 \cdot 2 = 8, \\ 2^{-5} &= (2^5)^{-1} = \frac{1}{2^5} = \frac{1}{2 \cdot 2 \cdot 2 \cdot 2 \cdot 2} = \frac{1}{32}. \end{aligned}$$

Theorem 2.25 (The Law of Exponents)**.** *If $x \neq 0$, $y \neq 0$ are real, then*

$$\left.\begin{array}{ll} \text{(i)} & (xy)^n = (x^n)(y^n), \\ \text{(ii)} & x^{m+n} = x^m \cdot x^n, \\ \text{(iii)} & x^{nm} = (x^n)^m, \end{array}\right\} \quad \textit{for all integers } m \textit{ and } n.$$

In particular, $x^{-n} = (x^{-1})^n$ for all real $x \neq 0$ and all integers n.

Proof. Conceptually, these results amount to counting the number of factors in a product. For instance, x^{m+n} represents a product of $(m+n)$ factors all equal to x; such a product can be separated into a product of m factors and a product of n factors, i.e., into $x^m \cdot x^n$. For illustration we give formal proofs by induction.

(Proof of (i)). For each natural number n, consider the statement

$$P(n): \qquad (xy)^n = (x^n)(y^n).$$

The base case $P(0)$ reduces to $1 = 1$, which is true. Assume inductively that $P(k)$ is true for some $k \geq 0$. We have

$$\begin{aligned} (xy)^{k+1} &= (xy)^k(xy) && \text{Definition of exponentiation,} \\ &= (x^k y^k)(xy) && \text{Inductive hypothesis,} \\ &= x^k(y^k \cdot x)y && \text{Associativity,} \\ &= x^k(x \cdot y^k)y && \text{Commutativity,} \\ &= (x^k \cdot x)(y^k \cdot y) && \text{Associativity,} \\ &= (x^{k+1})(y^{k+1}) && \text{Definition of exponentiation.} \end{aligned}$$

Since $P(0)$ is true and $P(k)$ implies $P(k+1)$ for all $k \geq 0$, the statement $P(n)$ is true for all $n \geq 0$ by the principle of mathematical induction. If $n < 0$, replace x and y by their multiplicative inverses.

Taking $y = x^{-1}$, we have

$$(x^n)(x^{-1})^n = 1^n = 1 = (x^n)(x^n)^{-1} = (x^n)(x^{-n});$$

by cancellation, $(x^{-1})^n = x^{-n}$ for all n.

(Proof of (ii)). We first assume m and n are non-negative integers. For each n in $\mathbf{N}$, consider the statement

$$P(n) \qquad x^{m+n} = x^m \cdot x^n \quad \text{for all } m \text{ in } \mathbf{N}.$$

This *single statement* may be viewed as an infinite family of statements, one for each natural number m, with n a fixed natural number.

The base case $P(0)$ asserts $x^{m+0} = x^m \cdot x^0$ for all m, which is true since $m + 0 = m$ for all m and $x^0 = 1$. Next, assume inductively that $P(k)$ is true for some k, namely that

$$x^{m+k} = x^m \cdot x^k \quad \text{for all } m \text{ in } \mathbf{N}.$$

For all m, we have

$$\begin{aligned} x^{m+k+1} &= x^{m+k} \cdot x && \text{Definition of exponentiation} \\ &= (x^m \cdot x^k) \cdot x && \text{Inductive hypothesis} \\ &= x^m \cdot (x^k \cdot x) && \text{Associativity} \\ &= x^m \cdot x^{k+1} && \text{Definition of exponentiation} \end{aligned}$$

which establishes the inductive step. By the principle of mathematical induction, $x^{m+n} = x^m \cdot x^n$ for all non-negative m and n.

If m and n are both non-positive, conclusion (ii) of the theorem now follows immediately by replacing x with x^{-1}.

It remains to check the case where one exponent is positive, the other negative. Without loss of generality, say $-m$ and n are positive.

Suppose first that $0 \leq m + n$. Since $-m$ is positive, the preceding argument shows

$$x^n = x^{(m+n)+(-m)} = x^{m+n} \cdot x^{-m}.$$

Multiplying both sides by x^m gives $x^{m+n} = x^m \cdot x^n$, as claimed.

If instead $m + n < 0$, then $0 < -(m + n)$, and

$$x^{-m} = x^{-(m+n)+n} = x^{-(m+n)} \cdot x^n;$$

rearranging establishes the asserted claim.

(Proof of (iii)). For m and n non-negative, this follows by induction on the statement

$$P(n) \qquad x^{nm} = (x^n)^m \quad \text{for all } m \text{ in } \mathbf{N}.$$

To establish the inductive step, note that

$$x^{n(m+1)} = x^{nm+n} = x^{nm} \cdot x^n = (x^n)^m \cdot x^n = (x^n)^{m+1}$$

by (ii) and the definition of exponentiation. As a fringe benefit, we find that

$$(x^m)^n = x^{mn} = x^{nm} = (x^n)^m \quad \text{for all } x \neq 0, \text{ all } m \text{ and } n \text{ in } \mathbf{N}.$$

If $m < 0$ or $n < 0$, replace x by x^{-1} and use $(x^{-1})^n = x^{-n}$. ☐

The "limiting behavior" of x^n as n grows without bound plays a central role in analysis.

Theorem 2.26. *Let $u > 0$ be a real number. We have*

$$1 + nu \leq (1+u)^n \quad \textit{for all } n \geq 0.$$

Proof. Let $P(n)$ denote the inequality in the theorem. The base case $P(0)$ asserts that $1 \leq 1$, which is true. Assume inductively that $P(k)$ is true for some $k \geq 0$. We have

$$\begin{aligned}
1 + (k+1)u &\leq 1 + (k+1)u + ku^2 && 0 \leq ku^2 \\
&= (1+ku)(1+u) && \text{Algebra} \\
&\leq (1+u)^k(1+u) && \text{Inductive hypothesis} \\
&= (1+u)^{k+1} && \text{Definition of exponentiation}
\end{aligned}$$

so that $P(k)$ implies $P(k+1)$. ☐

Corollary 2.27. *Let $0 < x < 1$. There exists a positive real number u such that $x = 1/(1+u)$, and*

$$0 < x^n \leq \frac{1}{1+nu} \quad \textit{for all } n \geq 0.$$

Proof. If $0 < x < 1$, then $1 < 1/x$ by Theorem 2.18 (v), so we may write $1/x = 1 + u$ for some positive real number u. By Theorem 2.26, we have $0 < 1 + nu \leq (1/x)^n = 1/(x^n)$ for all $n \geq 0$. Taking reciprocals again establishes the corollary. ☐

Factorials

Definition 2.28. Let $n \geq 0$ be an integer. We define the *factorial* of n recursively by

$$0! = 1, \qquad (n+1)! = (n+1) \cdot n! \quad \text{for } n \geq 0.$$

Example 2.29. Expanding the definition for $n = 5$, we have

$$5! = 5 \cdot 4! = 5 \cdot 4 \cdot 3! = 5 \cdot 4 \cdot 3 \cdot 2! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \cdot 0! = 120.$$

Generally, if $n > 0$, then $n!$ is the product of the positive integers from 1 up to n, inclusive. The first ten positive factorials, whose values are worth knowing, are $1! = 1$, $2! = 2$, $3! = 6$, $4! = 24$, $5! = 120$, $6! = 720$,

$$7! = 5040, \quad 8! = 40320, \quad 9! = 362880, \quad 10! = 3628800.$$

Lemma 2.30. *If n, $m \geq 0$ are integers, then $(n+m)! \geq (n+1)^m n!$. The inequality is strict if $m > 1$.*

Proof. Fix $n \geq 0$, and argue inductively on m. When $m = 0$ or 1, the inequality is an equality. Inductively, if $(n+k)! \geq (n+1)^k n!$ for some $k \geq 1$, then

$$\begin{aligned} (n+k+1)! &= (n+k+1) \cdot (n+k)! \\ &> (n+1) \cdot (n+k)! \\ &\geq (n+1) \cdot (n+1)^k n! = (n+1)^{k+1} n!, \end{aligned}$$

so strict inequality holds for $m = k+1$. □

Example 2.31. If $n \geq 0$, then (compare the case $m = 2$ in the lemma)

$$\frac{(2n)!}{(2n+2)!} = \frac{1}{(2n+2)(2n+1)} < \frac{1}{(2n+1)^2}.$$

2.5 The Triangle Inequalities

Definition 2.32. Let x be a real number. The *absolute value* of x is

$$|x| = \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0. \end{cases}$$

Remark 2.33. For all real x, $|-x| = |x|$. That is, signs may be factored from an absolute value without changing the expression.

Lemma 2.34. *If x is real, then $-|x| \leq x \leq |x|$.*

Proof. If $x \geq 0$, then $|x| = x$ and $-|x| = -x \leq 0$. Combining these inequalities, $-|x| \leq 0 \leq x \leq |x|$, as claimed.

If instead $x < 0$, then $|x| = -x$ and $0 < -x = |x|$. Consequently, $-|x| \leq x < 0 < |x|$. □

Proposition 2.35. *Let x and b denote real numbers.*

(i) $|x| < b$ *if and only if* $-b < x < b$.

(ii) *If* $|x| \leq b$ *for all* $b > 0$, *then* $x = 0$.

Proof. (i) By definition, $0 \leq |x|$ for all real x. If $|x| < b$, then Theorem 2.18 (iii) gives $-b < -|x| \leq 0 \leq |x| < b$. Combining with Lemma 2.34, we have $-b < -|x| \leq x \leq |x| < b$.

Conversely, suppose $-b < x < b$. If $0 \leq x$, then $x = |x|$, and we immediately deduce $|x| < b$. If $x < 0$, then $x = -|x|$, and by hypothesis we have $-b < x = -|x|$. Multiplying by -1 gives $|x| < b$ in this case as well.

(ii) The hypothesis involves infinitely many statements, one for each positive real number b, and *no finite number of these hypotheses implies the conclusion*. It is therefore more natural to consider the contrapositive, which turns out to be nearly obvious. If $x \neq 0$, then $0 < |x|$. Setting $b = |x|/2$, we have $0 < b < |x|$. □

We now establish two of the most ubiquitous inequalities in analysis, the *triangle inequality* and *reverse triangle inequality*.

Theorem 2.36. *If x and y are real numbers, then*

(i) $|x + y| \leq |x| + |y|$.

(ii) $|x - y| \geq \big||x| - |y|\big|$.

Remark 2.37. In fact, $|x \pm y| \leq |x| + |y|$ and $|x \pm y| \geq \big||x| - |y|\big|$ for all real x and y, since $|-y| = |y|$.

Proof. (i) For arbitrary real x, we have $-|x| \leq x \leq |x|$ by Lemma 2.34, and similarly $-|y| \leq y \leq |y|$. Adding these inequalities and using (iii),

$$-\big(|x| + |y|\big) \leq x + y \leq |x| + |y|,$$

which is equivalent to $|x + y| \leq |x| + |y|$ by Proposition 2.35.

(ii) Apply (i) to the identity $y = x + (y - x)$, obtaining

$$|y| \leq |x| + |y - x|, \quad \text{or} \quad |y| - |x| \leq |y - x|.$$

Reversing the roles of x and y shows $|x| - |y| \leq |x - y|$, and therefore $|y| - |x| \geq -|x - y| = -|y - x|$. Combining,

$$-|y - x| \leq |y| - |x| \leq |y - x|.$$

By Proposition 2.35, $\big||y| - |x|\big| \leq |y - x|$. □

Definition 2.38. Let x and y be real numbers. The *distance* between x and y is

$$|y - x| = |x - y| = \begin{cases} y - x & \text{if } x \leq y, \\ x - y & \text{if } y < x. \end{cases}$$

Corollary 2.39. *If x, y and z are real numbers, then*

(i) $|x - z| \leq |x - y| + |z - y|$.

(iii) $|x - z| \geq \big||x - y| - |z - y|\big|$.

Remark 2.40. These claims follow immediately by applying the corresponding part of Theorem 2.36 to the identity

$$(x - z) = (x - y) - (z - y).$$

We may view real numbers x, y, and z as vertices of a (degenerate) triangle on the number line. Part (i) of the corollary says the length of a side of a triangle does not exceed the sum of the lengths of the other two sides. Part (ii) says the length of a side of a triangle is no shorter than the difference of the lengths of the other two sides.

Min and Max

Definition 2.41. Let a and b be real numbers. We define their *maximum* and *minimum* by

$$\max(a, b) = \begin{cases} b & \text{if } a \leq b, \\ a & \text{if } b < a, \end{cases} \qquad \min(a, b) = \begin{cases} a & \text{if } a \leq b, \\ b & \text{if } b < a. \end{cases}$$

Proposition 2.42. *If a and b are real numbers, then*

$$\max(a, b) = \frac{a + b + |a - b|}{2}, \qquad \min(a, b) = \frac{a + b - |a - b|}{2}.$$

Proof. Adding and subtracting the formulas for $\max(a, b)$ and $\min(a, b)$ in Definition 2.41 gives

$$\begin{aligned} \max(a, b) + \min(a, b) &= a + b \\ \max(a, b) - \min(a, b) &= |a - b| = \begin{cases} b - a & \text{if } a \leq b, \\ a - b & \text{if } b < a. \end{cases} \end{aligned}$$

The proposition follows by adding and subtracting *these* and dividing by 2, compare Exercise 2.4. □

Proposition 2.43. *If* $A = \{a_1, a_2, \dots, a_n\}$ *is a* finite *set of real numbers, there exist unique elements* $\max A$ *and* $\min A$ *in* A *such that*

$$\min A \leq x \leq \max A \quad \textit{for all } x \textit{ in } A.$$

Remark 2.44. We call $\max A = \max(a_1, a_2, \dots, a_n)$ the *largest* element of A and $\min A = \min(a_1, a_2, \dots, a_n)$ the *smallest* element of A.

Proof. (Existence). If $A = \{a, b\}$ is a set of one or two elements, Definition 2.41 defines $\min A$ and $\max A$. Inductively, define

$$\max(a_1, a_2, \dots, a_n) = \max\bigl(\max(a_1, a_2, \dots, a_{n-1}), a_n\bigr),$$

and check that $x \leq \max A$ for all x in A. The smallest element is defined similarly.

(Uniqueness). If α and α' are elements of A satisfying $x \leq \alpha$ and $x \leq \alpha'$ for all x in A, then $\alpha \leq \alpha'$ (since $\alpha \in A$) and $\alpha' \leq \alpha$ (since $\alpha' \in A$), so $\alpha = \alpha'$. Uniqueness of $\min A$ is proven similarly. □

2.6 Real Arithmetic

The real number line and base-10 notation for real numbers are probably so familiar that it is difficult even to consider analyzing them closely. This brief, relatively intuitive section makes contact between the familiar number line and decimal notation of school mathematics on one hand, and our axioms for the real number system on the other.

The Number Line

Thanks to the order axioms, real numbers may be viewed as points on a line; $x < y$ if and only if the point x lies to the left of y. The real numbers 0 and 1 determine an *origin* and *length scale*. Two real numbers x and $x+1$ are separated by “unit distance”. In particular, the set of integers is situated along the number line with adjacent integers separated by unit distance.

It will follow from the “Archimedean property” (Theorem 3.56) that every real number x can be written *uniquely* as an integer $\lfloor x \rfloor$ plus a non-negative real number $x - \lfloor x \rfloor < 1$. If $0 \leq x$, these are called, respectively, the *integer part* and *fractional part* of x.

Rational Numbers

Let p and q be integers, $q \neq 0$. The "rational number" p/q is the real number $r = pq^{-1}$, the product of p and the multiplicative inverse of q. Procedurally, $1/q$ represents a quantity, q portions of which make 1, and p/q represents p units of size $1/q$. The number p is the *numerator*; it specifies how many portions r comprises. The number q is the *denominator*; it specifies the size of each portion comprising r.

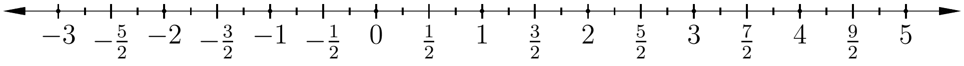


Figure 2.1: Rational numbers on the number line.

Remark 2.45. Three portions of size $1/3$ make 1, as do 42 portions of size $1/42$. Making $q > 0$ larger increases the number of portions required to make 1, so $1/q$ decreases in size when $q > 0$ increases.

This is an example of Theorem 2.18 (v). It is also the basis of Yogi Berra's quip: Asked by a waiter, "Do you want your pizza cut into six or eight pieces?", Berra responded, "Better make it six; I could never eat eight pieces of pizza."

To add two rational numbers, we agglomerate amounts, first expressing each summand as a fraction with a common denominator, then adding the numerators. To multiply rational numbers r and s, we regard s as a unit, of which a portion r is to be taken.

Example 2.46. If $r = 1/2$ and $s = 2/3$, each summand can be represented as a whole number of portions of size $1/6$, so

$$r + s = \frac{1}{2} + \frac{2}{3} = \frac{3}{6} + \frac{4}{6} = \frac{7}{6}.$$

To multiply, we have

$$rs = \frac{1}{2} \cdot \frac{2}{3} = \frac{1}{3},$$

since 2 portions of size $1/3$ make $2/3$.

Proposition 2.47. *If $r = p_1/q_1$ and $s = p_2/q_2$ are rational numbers and $k \neq 0$ is an integer, then*

(i) $r = (kp_1)/(kq_1)$.

(ii) $r + s = \dfrac{p_1q_2 + q_1p_2}{q_1q_2}$.

(iii) $rs = \dfrac{p_1p_2}{q_1q_2}$.

Proof. (i) Freely using the associative and commutative axioms for multiplication,

$$(kp_1)/(kq_1) = kp_1(k^{-1})(q_1^{-1}) = k(k^{-1})p_1(q_1^{-1}) = p_1(q_1^{-1}) = p_1/q_1.$$

(ii) Using (i), we have

$$\frac{p_1}{q_1} + \frac{p_2}{q_2} = \frac{p_1q_2}{q_1q_2} + \frac{q_1p_2}{q_1q_2} = \frac{p_1q_2 + q_1p_2}{q_1q_2}.$$

(iii) Since $(q_1q_2)^{-1} = (q_1^{-1})(q_2^{-1})$, we have

$$\frac{p_1}{q_1} \cdot \frac{p_2}{q_2} = (p_1)(q_1^{-1})(p_2)(q_2^{-1}) = (p_1p_2)(q_1q_2)^{-1} = \frac{p_1p_2}{q_1q_2}. \qquad \square$$

Remark 2.48. These identities hold (with identical proofs) if the numbers p_1, $q_1 \neq 0$, ..., are arbitrary *real* numbers, not merely integers.

Corollary 2.49. *The set* $\mathbf{Q}$ *of rational numbers is closed under addition and under multiplication. If* $r \in \mathbf{Q}$*, then* $-r \in \mathbf{Q}$*. If in addition* $r \neq 0$*, then* $r^{-1} = 1/r \in \mathbf{Q}$*.*

Irrational Numbers

Proposition 2.50. *If* r *is rational and* x *is irrational, then* $r + x$ *is irrational. If* $r \neq 0$*, then* rx *is irrational.*

Proof. Contrapositively, if $r+x = s$ were rational, then $x = s-r$ would be rational as well. If $r \neq 0$ but $t = rx$ were rational, then similarly, $x = t/r$ would be rational. $\square$

Remark 2.51. A sum or a product of irrational numbers can be either rational or irrational. Granting that every positive integer has a real square root, and that the square root of an integer is irrational unless the integer is a perfect square, $\sqrt{2} + (-\sqrt{2}) = 0$ is rational, while $\sqrt{2}+\sqrt{2} = 2\sqrt{2} = \sqrt{8}$ is irrational. Similarly, $(\sqrt{2})(\sqrt{2}) = 2$ is rational, but $(\sqrt{2})(\sqrt{3}) = \sqrt{6}$ is irrational.

Decimal Representation

Electronic computers perform exact integer arithmetic: If the operands and result do not overflow the machine's native "word size" (at this writing, usually $2^{64} - 1 = 18446744073709551615$), then arithmetic is not subject to round-off error, and the associative, commutative, and distributive laws hold. A computer can be programmed to perform exact rational arithmetic, since a rational number is effectively an ordered pair of integers.

However, most numerical work in a computer uses "floating point" arithmetic, which is not exact, i.e., does not obey the associative, commutative, or distributive laws. Thanks to the ubiquity of electronic calculators in the schools, you are probably much more familiar with floating point arithmetic and the associated "decimal notation" than with exact rational arithmetic.

A finite decimal expression such as 3.14159 stands for a rational number:

$$3 + \frac{1}{10} + \frac{4}{10^2} + \frac{1}{10^3} + \frac{5}{10^4} + \frac{9}{10^5} = \frac{314,159}{100,000}.$$

The denominator of such an expression may be taken to be a power of ten. A rational number whose denominator does not evenly divide some power of ten can only be represented as an infinite decimal, albeit a decimal that eventually repeats after finitely many terms:

$$\frac{1}{3} = 0.\overline{3}, \qquad \frac{2}{11} = 0.\overline{18}, \qquad \frac{7}{12} = 0.58\overline{3}.$$

(A bar over a group of digits denotes an infinitely repeating pattern.) A careful proof that a non-terminating rational decimal p/q eventually repeats is slightly involved, but the idea is not difficult. We may write $p = mq + r$ uniquely with m and r integers and $1 \leq r < q$. (If $r = 0$, then $p/q = m$ is an integer.) Use the long division algorithm to divide r by q. The remainder at each step can only be an integer between 1 and $q - 1$. (If the remainder were 0, the decimal would terminate after finitely many places.) Since zeros are brought down at each step, only finitely many trial divisions arise. As soon as some remainder is seen a second time, the subsequent remainders repeat cyclically *ad infinitum.*

Conversely, a repeating decimal expression always represents a rational number. Again, a general proof is a bit involved notationally, but the idea is quite simple, and well-illustrated by an example: If

$x = 3.14\overline{159}$, then we have

$$\begin{aligned} 100000x &= 314159.\overline{159}, \\ 100x &= 314.\overline{159}, \\ 99900x &= 313845, \quad \text{or} \quad x = \frac{313845}{99900} = \frac{20923}{6660}. \end{aligned}$$

Why Real Numbers?

Exact rational arithmetic is simple and explicit, any two rational numbers can be compared, and there exist arbitrarily small rational numbers, such as 0.1, 0.01, 0.001, etc. Why not use rational numbers as the basis of analysis?

In hindsight, the rational numbers have a serious defect. Intuitively, there exist quantities that we'd like to regard as "numbers" (because they can be approximated as closely as we like by fractions), but the quantities themselves are not rational numbers.

The classic example is $\sqrt{2}$, the hypotenuse of a right isosceles triangle with unit legs. There is no rational number $x = p/q$ satisfying $x^2 = 2$. Nonetheless, $\sqrt{2}$ "exists" as a geometric length, and we can construct a sequence of rational numbers whose squares get "closer and closer" to 2; intuitively, we can approximate $\sqrt{2}$ as closely as we like by rational numbers.

Exercises

Exercise 2.1. Without using a calculator, express each of the fractions $\frac{1}{7}$, $\frac{2}{7}$, $\frac{3}{7}$, $\frac{4}{7}$, $\frac{5}{7}$, and $\frac{6}{7}$ as repeating decimals.
Suggestion: You'll need to do long division at least once, but can reduce the amount of computation by exploiting patterns.

Exercise 2.2. Express $1.\overline{12}$, $0.24\overline{9}$, and $0.041\overline{6}$ as reduced fractions.

Exercise 2.3. Let x and y be real. Prove that $xy = \frac{1}{2}\big((x+y)^2 - x^2 - y^2\big)$. (In words, multiplication is uniquely determined by squaring.)

Exercise 2.4. Let x, y, u, and v be real. Prove that $2u = x + y$ and $2v = x - y$ if and only if $x = u + v$ and $y = u - v$.

Exercise 2.5. Let x and y be real. Prove that $x^2 = y^2$ if and only if $x = y$ or $x = -y$. (You may use the axioms and results from the text. Do not use properties of square roots, which have not yet been established.)

Exercise 2.6. Alice and Bob have two sandwiches. Because Alice isn't very hungry, they will split one sandwich evenly, and Bob will eat the other.

(a) In order to distribute the food most nearly equally, should they split the smaller or the larger sandwich?

(b) Formulate a precise mathematical model for the sandwiches, and use your model to give a proof of your assertion in part (a).

Exercise 2.7. Let a, b, and c be real numbers, and assume $a > 0$. By completing the square, show that $ax^2 + bx + c \geq (4ac - b^2)/(4a)$ for all real x, with equality if and only if $x = -b/(2a)$.

Exercise 2.8. Let $n > 0$ be an odd integer. Prove that if $x < y$ are real numbers, then $x^n < y^n$.

Exercise 2.9. Give a formal proof by mathematical induction that if x_1, x_2, …, x_n are real numbers, then

$$|x_1 + x_2 + \cdots + x_n| \leq |x_1| + |x_2| + \cdots + |x_n|.$$

Exercise 2.10. Let $u > 0$ be real. Use induction to prove that

$$1 + nu + \frac{n(n-1)}{2}\,u^2 \leq (1+u)^n \quad \text{for all } n \geq 0.$$

Exercise 2.11. Let x and y be real numbers, and assume $x < y$.

(a) Prove that $x < \frac{1}{2}(x+y) < y$ and $x < \frac{1}{3}(2x+y) < \frac{1}{3}(x+2y) < y$.

(b) More generally, show that if $0 < t < 1$, then $x < (1-t)x + ty < y$. (The expression $(1-t)x + ty$ is called a *convex linear combination* of x and y.)

(c) If $s < t$ are real, then $(1-s)x + sy < (1-t)x + ty$.

Exercise 2.12. Let x_0 and $r > 0$ be real numbers. Prove that if x is real, then

(a) $|x - x_0| < r$ if and only if $x_0 - r < x < x_0 + r$.

(b) $0 < |x - x_0| < r$ if and only if $x_0 - r < x < x_0$ or $x_0 < x < x_0 + r$.

Chapter 3

Sets of Real Numbers

3.1 Intervals

Definition 3.1. Let a and b be real numbers with $a < b$. The sets

$$\begin{aligned}(a, b) &= \{x \text{ in } \mathbf{R} : a < x < b\}\\ [a, b] &= \{x \text{ in } \mathbf{R} : a \le x \le b\}\end{aligned}$$

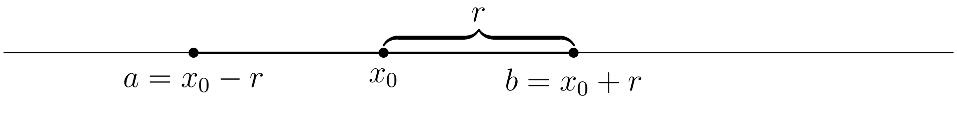
are called, respectively, the *open interval* and the *closed interval* with endpoints a and b.

Remark 3.2. There are also *half-open* intervals

$$[a, b) = \{x \text{ in } \mathbf{R} : a \le x < b\}, \qquad (a, b] = \{x \text{ in } \mathbf{R} : a < x \le b\},$$

though these play a relatively minor role in elementary analysis.

Remark 3.3. For any of these intervals, the midpoint $x_0 = \frac{1}{2}(b + a)$ is called the *center*, $r = \frac{1}{2}|b - a|$ is the *radius*, and $|b - a| = 2r$ the *length*.



The formulas for $\min(a, b)$ and $\max(a, b)$ given by Proposition 2.42 have geometric interpretations. These numbers are the left- and right-hand endpoints of the closed interval $[a, b]$, and are reached by starting at the midpoint x_0 and traveling a distance r to the left or right.

Definition 3.4. Let x_0 be a real number, and r a positive real number. When we wish to emphasize the center and radius of an interval, we use the notation

$$B_r(x_0) = (x_0 - r, x_0 + r),$$

and call this set the *(open) ball of radius r about x_0*.

Proposition 3.5. *Let a and b be real numbers such that $a < b$, and set $x_0 = \frac{1}{2}(b + a)$ and $r = \frac{1}{2}|b - a|$. Then*

$$(a, b) = \{x \text{ in } \mathbf{R} : |x - x_0| < r\}, \qquad [a, b] = \{x \text{ in } \mathbf{R} : |x - x_0| \leq r\}.$$

Proof. We have $a = x_0 - r$ and $b = x_0 + r$. If x is real, then $x \in (a, b)$ if and only if $a < x < b$, i.e., $x_0 - r < x < x_0 + r$. Subtracting x_0 from each term gives $-r < x - x_0 < r$, and by Proposition 2.35 this is equivalent to $|x - x_0| < r$. Replacing the "$<$"s with "$\leq$" proves the assertion about the closed interval. □

Remark 3.6. Open balls are nested by transitivity of inequality: If $r' < r$, then $B_{r'}(x_0) \subseteq B_r(x_0)$. This observation is of fundamental importance in analysis.

Remark 3.7. As we have seen in Chapter 2, two distinct real numbers are not "neighbors"; if $a < b$, there is a "gulf" (a, b) separating a and b. In order to bridge this gulf, we must consider "neighbors" of x_0 not as individual numbers, but as open intervals.

Definition 3.8. Let a be a real number. We define *unbounded* open and closed intervals by

$$\begin{aligned} (-\infty, a) &= \{x \text{ in } \mathbf{R} : x < a\} & \qquad (a, \infty) &= \{x \text{ in } \mathbf{R} : a < x\}, \\ (-\infty, a] &= \{x \text{ in } \mathbf{R} : x \leq a\} & \qquad [a, \infty) &= \{x \text{ in } \mathbf{R} : a \leq x\}. \end{aligned}$$

Remark 3.9. The symbols $-\infty$ and ∞ do not denote real numbers, but are merely place-holders for an omitted inequality.

Example 3.10. If $a < b$, then $(a, b) = (-\infty, b) \cap (a, \infty)$, etc.

Arithmetic of Intervals

Analysis in one real variable deals not merely with individual numbers, but with intervals of numbers. We introduce special notation that facilitates calculations with intervals.

Definition 3.11. Let x_0 be real, $r > 0$. We write $x \approx x_0 + \mathsf{A}(r)$, read "$x$ is x_0 plus A of r", to signify that $|x - x_0| < r$, namely that x is approximately equal to x_0 up to an error of absolute value less than r.

Remark 3.12. In other words, $x \approx x_0 + \mathsf{A}(r)$ if and only if $x \in B_r(x_0)$, if and only if $x_0 - r < x < x_0 + r$. In this interpretation of an open ball, we call x_0 the *value* and r the *error.*

Each of these three notations facilitates useful idioms, so it's worthwhile to become comfortable with each.

Remark 3.13. We agree that $x \approx x_0 + \mathsf{A}(0)$ means "$x \approx x_0 + \mathsf{A}(r)$ for every $r > 0$", namely that $x = x_0$ in the usual sense.

Remark 3.14. In the sciences, the "$\approx$" symbol is often used between two real expressions that are not exactly equal, but "are nearly enough equal for the sake of argument", as in "$\sin\theta \approx \theta$ if $\theta \approx 0$". Here we are giving the symbol precise meaning in a manner amenable to algebraic manipulation.

Example 3.15. Experimental data and other empirical measurements always come with uncertainties.

If the temperature T at noon at the airport was 74° F, to within 0.5°, we could write $T \approx 74 + \mathsf{A}(0.5)$.

If a bacterial culture has density ρ of $1,000,000$ per square millimeter, give or take $100,000$, we might write $\rho \approx 10^6 + \mathsf{A}(10^5)$.

If a baseball has a mass m of 143 grams with an uncertainty of 0.05 grams, we could express this as $m \approx 143 + \mathsf{A}(0.05)$.

Definition 3.16. Let $M > 0$. We write $x_0 + \mathsf{A}(r) \approx \mathsf{A}(M)$ to signify that if $|x - x_0| < r$, then $|x| < M$.

Example 3.17. Smaller errors correspond to "better" (more detailed) information. For example,

$$\begin{aligned}\sqrt{2} &\approx 1.4142136 + \mathsf{A}(5 \times 10^{-6}) \\ &\approx 1.414 + \mathsf{A}(0.0003) \\ &\approx 1.4 + \mathsf{A}(0.02).\end{aligned}$$

Theorem 3.18. *Let x_0 and y_0 be real numbers, and let r and s be non-negative. If $x \approx x_0 + \mathsf{A}(r)$ and $y \approx y_0 + \mathsf{A}(s)$, then*

(i) *$x + y \approx x_0 + y_0 + \mathsf{A}(r + s)$.*

(ii) *$xy \approx x_0y_0 + A\big(|x_0|s + r|y_0| + rs\big)$.*

(iii) *If in addition $x_0 + \mathsf{A}(r) \approx \mathsf{A}(M)$ and $y_0 + \mathsf{A}(s) \approx \mathsf{A}(N)$ for some positive real numbers M and N, then*

$$xy \approx x_0y_0 + A\big(Ms + Nr\big).$$

Proof. Each is clear if $r = 0$ or $s = 0$. Assume r and s are positive.

(i) By hypothesis, $x_0 - r < x < x_0 + r$ and $y_0 - s < y < y_0 + s$. Adding these inequalities,

$$(x_0 + y_0) - (r + s) < x + y < (x_0 + y_0) + (r + s).$$

(ii) Writing $x = x_0 + (x - x_0)$ and $y = y_0 + (y - y_0)$, we have

$$\begin{aligned} xy - x_0y_0 &= \big(x_0 + (x - x_0)\big)\big(y_0 + (y - y_0)\big) - x_0y_0 \\ &= x_0(y - y_0) + (x - x_0)y_0 + (x - x_0)(y - y_0). \end{aligned}$$

The triangle inequality gives

$$\begin{aligned} |xy - x_0y_0| &\leq |x_0|\,|y - y_0| + |x - x_0|\,|y_0| + |x - x_0|\,|y - y_0| \\ &< |x_0|s + r|y_0| + rs. \end{aligned}$$

(iii) If $x_0 + \mathsf{A}(r) \approx \mathsf{A}(M)$ and $y_0 + \mathsf{A}(s) \approx \mathsf{A}(N)$, then by (ii),

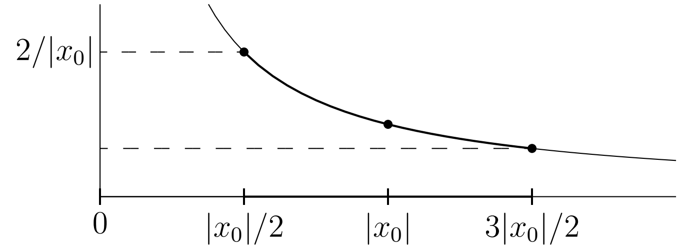
$$|xy - x_0y_0| < |x_0|s + r|y_0| + rs = \big(|x_0| + r\big)s + r|y_0| < Ms + Nr. \qquad \square$$

Example 3.19. If $x \approx x_0 + \mathsf{A}(0.1)$, we have $-3x \approx -3x_0 + \mathsf{A}(0.3)$, $-3x + 5 \approx -3x_0 + 5 + \mathsf{A}(0.3)$, and $x^2 \approx x_0^2 + \mathsf{A}(0.2|x_0| + 0.01)$.

Proposition 3.20. *Let $x_0 \neq 0$. If $x \approx x_0 + \mathsf{A}(r)$ for some $r \leq |x_0|/2$, then*

$$\frac{1}{x} \approx \frac{1}{x_0} + \mathsf{A}\left(\frac{2r}{|x_0|^2}\right).$$

Proof. If $x \approx x_0 + \mathsf{A}(r)$ and $r \leq |x_0|/2$, then $|x - x_0| < r \leq |x_0|/2$, or $-|x - x_0| > -|x_0|/2$. Since $x = x_0 - (x_0 - x)$, the reverse triangle inequality implies $|x| \geq |x_0| - |x_0 - x| > |x_0|/2$.



That is, if $x \approx x_0 + \mathsf{A}(r)$ for some $r \leq |x_0|/2$, then $|x| > |x_0|/2 > 0$, so $1/|x| < 2/|x_0|$, and consequently

$$\left|\frac{1}{x} - \frac{1}{x_0}\right| = \frac{|x - x_0|}{|x|\,|x_0|} < \frac{2|x - x_0|}{|x_0|^2} < \frac{2r}{|x_0|^2}. \qquad \square$$

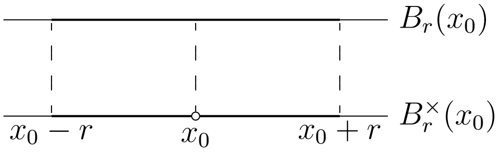
3.2 Operations on Sets

In Chapter 1, we encountered the complement of a set, the difference of two sets, and the union or intersection or an arbitrary family of sets. Starting from closed and open intervals, these operations can be used to construct interesting sets of real numbers, or to express interesting properties of the real number system.

Example 3.21. Let x_0 be a real number, and let $r > 0$. We define the *deleted ball* $B_r^\times(x_0)$ of radius r about x_0 to be the set

$$\begin{aligned} B_r(x_0) \setminus \{x_0\} &= (x_0 - r, x_0) \cup (x_0, x_0 + r) \\ &= \{x \text{ in } \mathbf{R} : 0 < |x - x_0| < r\}. \end{aligned}$$

That is, $B_r^\times(x_0)$ is the ball $B_r(x_0)$ from which the center x_0 has been removed.



Example 3.22. For all real a, we have $(-\infty, a) = \mathbf{R} \setminus [a, \infty)$, etc.

Example 3.23. Let a be a real number. The one-element set $\{a\}$ is called a *singleton.* Every set can be written as the union of its singleton subsets:

$$A = \bigcup_{a \in A} \{a\}.$$

Definition 3.24. Assume $A \subseteq \mathbf{R}$, and let c be a real number. The set

$$c + A = \{c + a : a \in A\}$$

is called the *translation* of A by c. The set

$$cA = \{ca : a \in A\}$$

is called the *scaling* of A by c. In particular, $-A = \{-a : a \in A\}$ is the reflection of A across the origin.

Remark 3.25. If $c = 0$, then $c + A = A$ and $cA = \{0\}$. In practice, we almost always assume $c \neq 0$.

Example 3.26. If A and B are non-empty sets of real numbers, we define their *sum* and *product* to be

$$A + B = \{a + b : a \in A \text{ and } b \in B\} = \bigcup_{a \in A} a + B,$$
$$AB = \{ab : a \in A \text{ and } b \in B\} = \bigcup_{a \in A} aB.$$

Particularly, $\mathbf{Z} + [0, 1) = \mathbf{R}$, $\mathbf{Z} + \mathbf{Z} = \mathbf{Z}$, and $\mathbf{Q} + \mathbf{Q} = \mathbf{Q}$.

Example 3.27. Let $q \geq 1$ be an integer. The set

$$\tfrac{1}{q}\mathbf{Z} = \{p/q \text{ in } \mathbf{R} : p \in \mathbf{Z}\}$$

consists of all rational numbers that can be represented as a (possibly improper and/or non-reduced) fraction whose denominator is *precisely* q.

The elements of $\frac{1}{q}\mathbf{Z}$ are spaced regularly along the number line, with adjacent elements separated by a distance of $1/q$, Figure 3.1.

If q and q' are positive, then $\frac{1}{q}\mathbf{Z} \subseteq \frac{1}{q'}\mathbf{Z}$ if and only if q divides q'.

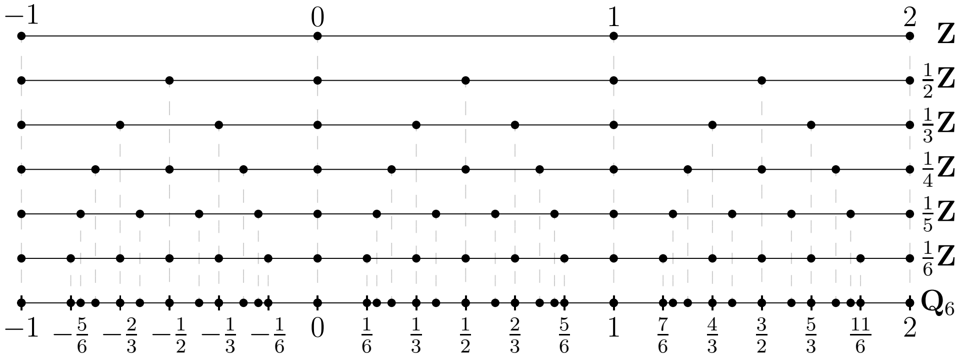


Figure 3.1: The sets $\frac{1}{q}\mathbf{Z}$ and $\mathbf{Q}_N$.

Example 3.28. Let $N \geq 1$ be an integer. The set

$$\mathbf{Q}_N = \bigcup_{q=1}^{N} \tfrac{1}{q}\mathbf{Z}$$

consists of all rational numbers that can be represented as a (possibly improper and/or non-reduced) fraction whose denominator is *no larger than* N.

The sets $\mathbf{Q}_N$ are *nested outward*: $\mathbf{Q}_N \subseteq \mathbf{Q}_{N+1} = \mathbf{Q}_N \cup \frac{1}{N+1}\mathbf{Z}$.

Example 3.29. With the notation of Examples 3.27 and 3.28, the set of rational numbers can be expressed in two ways as an infinite union:

$$\mathbf{Q} = \bigcup_{q=1}^{\infty} \tfrac{1}{q}\mathbf{Z} = \bigcup_{N=1}^{\infty} \mathbf{Q}_N.$$

Example 3.30 (The Cantor ternary set)**.** Let $[a, b]$ be an arbitrary closed interval. The weighted averages

$$a = \tfrac{1}{3}(3a) < \tfrac{1}{3}(2a + b) < \tfrac{1}{3}(a + 2b) < \tfrac{1}{3}(3b) = b$$

subdivide the interval into three pieces of equal length. The result of "removing the (open) middle third" of $[a, b]$ is the set

$$[a, b]\check{} = [a, b] \setminus \big(\tfrac{1}{3}(2a + b), \tfrac{1}{3}(a + 2b)\big) = \big[a, \tfrac{1}{3}(2a + b)\big] \cup \big[\tfrac{1}{3}(a + 2b), b\big].$$

The closed intervals $\big[a, \frac{1}{3}(2a + b)\big]$ and $\big[\frac{1}{3}(a + 2b), b\big]$ will be called the *components* of $[a, b]\check{}$ in this context.

Recursively construct a family of sets as follows. Let $K_0 = [0, 1]$ be the closed unit interval. Then let K_1 be the result of removing the middle third of K_0, let K_2 be the result of removing the middle third of each component of K_1, and generally, let K_{n+1} be the result of removing the middle third of each component of K_n.

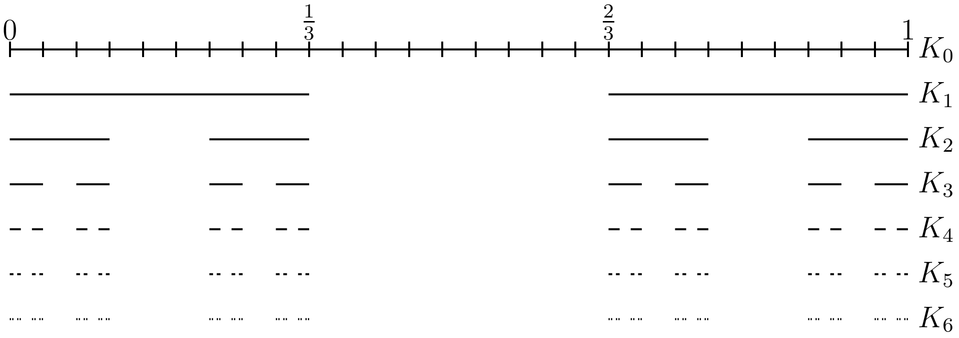


Figure 3.2: Approximations to the Cantor ternary set.

Mathematical induction shows the set K_n is a union of 2^n components, each of length 3^{-n}. The sets K_n are *nested inward*: $K_n \supseteq K_{n+1}$ for each $n \geq 0$.

The intersection

$$K = \bigcap_{n=0}^{\infty} K_n$$

is called the *Cantor ternary set*, or simply "the Cantor set". Each endpoint of K_n is an element of K_{n+1}, and consequently "survives" in the intersection. The Cantor set therefore contains the union of the endpoints of the sets K_n. Since K_n has 2^{n+1} endpoints, K has infinitely many elements. (K also contains "non-endpoint" elements.)

The Cantor set is *self-similar*. Precisely, $K = \frac{1}{3}K \cup \frac{1}{3}(K + 2)$; the Cantor set is a union of two disjoint subsets, each a scaled copy one-third the size of the entire set.

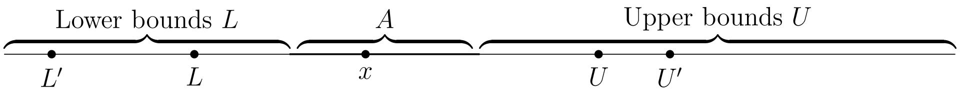
3.3 Upper and Lower Bounds

Definition 3.31. Let A be a set of real numbers. A real number U is an *upper bound* of A if $x \leq U$ for every x in A. If there exists an upper bound of A, we say A is *bounded above* (in $\mathbf{R}$).

A real number L is a *lower bound* of A if $L \leq x$ for every x in A. If there exists a lower bound of A, we say A is *bounded below* (in $\mathbf{R}$).

The set A is *bounded* if A is bounded above and bounded below.

Remark 3.32. If U is an upper bound of A and if $U < U'$, then U' is an upper bound of A by transitivity of inequality. Similarly, if L is a lower bound of A and if $L' < L$, then L' is also a lower bound of A.



Example 3.33. Let $a < b$ be real numbers. The interval $[a, b]$ is bounded. The left-hand endpoint a is a lower bound, and the right-hand endpoint b is an upper bound.

Every subset of a bounded set is obviously bounded. For example, the open interval (a, b) and the Cantor set $K \subseteq [0, 1]$ are bounded.

Remark 3.34. If $A \subseteq \mathbf{R}$, then the following are equivalent:

(i) L is a lower bound of A and U is an upper bound of A.

(ii) $A \subseteq [L, U]$.

Example 3.35. The set $\mathbf{N}$ of natural numbers is bounded below; 0 is a lower bound. As we will see presently (Theorem 3.56), $\mathbf{N}$ is not bounded above in **R**. In other words, for every real number x, there exists a natural number n such that $x < n$.

The set of integers is not bounded above or below, nor is any set that contains **Z**; thus **Q** and **R** are not bounded above or below.

Proposition 3.36. *A subset A of* **R** *is bounded if and only if there exists a positive real number M such that $|x| \le M$ for all x in A.*

Proof. If there exists an $M > 0$ such that $|x| \le M$ for all x in A, then $-M \le x \le M$ for all x in A, i.e., $L = -M$ and $U = M$ are lower and upper bounds of A, respectively.

Conversely, if $L \le x \le U$ for all x in A, take $M = 1 + \max\big(|L|, |U|\big)$. For all x in A, we have $-M < |L| \le L \le x \le U \le |U| < M$. $\square$

Remark 3.37. In practice, we often choose "symmetrical" upper and lower bounds for a bounded set.

The Completeness Axiom

Definition 3.38. Let A be a non-empty set of real numbers. A real number M in A is called the *maximum* of A or the *largest element* of A if $x \le M$ for every x in A.

A real number m in A is called the *minimum* of A or the *smallest element* of A if $m \le x$ for every x in A.

Remark 3.39. We often write $M = \max A$ and $m = \min A$. If a set A *has* a largest element $\max A$, then A is bounded above by $\max A$. Contrapositively, if A is not bounded above, then A has no largest element. Analogous remarks hold for lower bounds and smallest elements.

Note carefully that a bounded, non-empty set A *need not have* a maximum element or minimum element, see Example 3.42.

Example 3.40. A finite set of real numbers $A = \{a_1, a_2, \dots, a_n\}$ has a maximum and a minimum, see Proposition 2.43.

Example 3.41. A closed interval $[a, b]$ has a smallest element a and a largest element b.

Example 3.42. Let $a < b$ be real numbers. The open interval (a, b) is bounded below by a and bounded above by b, but contains no smallest or largest element: If $x \in (a, b)$, i.e., if $a < x < b$, then by Exercise 2.11,

$$a < \tfrac{1}{2}(a + x) < x < \tfrac{1}{2}(x + b) < b.$$

In words, $\frac{1}{2}(a+x)$ in (a, b) is smaller than x (so x is not the minimum), and $\frac{1}{2}(x+b)$ in (a, b) is larger.

Definition 3.43. Let A be a set of real numbers that is bounded above. A real number β is called a *least upper bound* or *supremum* of A if

(i) $x \leq \beta$ for all x in A, i.e., β is an upper bound of A.

(ii) For every upper bound U of A, we have $\beta \leq U$.

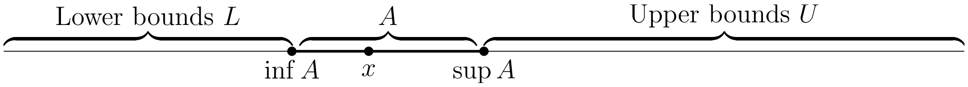
Lemma 3.44. *If $A \subseteq \mathbf{R}$, and β and β' are suprema of A, then $\beta = \beta'$.*

Proof. By hypothesis, β' is an upper bound of A, so $\beta \leq \beta'$ by condition (ii). Reversing roles, $\beta' \leq \beta$. □

Remark 3.45. We are therefore justified in writing $\sup A$ to denote the least upper bound of A; the symbol $\sup A$ may signify no number at all, but it never signifies more than one.

The completeness axiom says that if A is a *non-empty* set of real numbers that is bounded above, then $\sup A \in \mathbf{R}$.

If A has a largest element, then $\sup A = \max A$. Otherwise, $\sup A$ is the leftmost number lying to the right of every element of A.



Proposition 3.46. *Let A be a set of real numbers having a least upper bound β. Then β is the unique number satisfying the conditions*

(i) *If $x \in A$, then $x \leq \beta$.*

(ii)′ *For every $\varepsilon > 0$, there exists an x in A such that $\beta - \varepsilon < x$.*

Proof. Conceptually, the conditions (ii) and (ii)′ are contrapositives.

In other words, β is the least upper bound of A if and only if every upper bound U of A satisfies $\beta \leq U$. The contrapositive says that for every $U < \beta$, U *fails* to be an upper bound of A. That is, for every $\varepsilon > 0$, there exists an x in A such that $U = \beta - \varepsilon < x$. □

Everything said above for upper bounds has a corresponding concept or statement for lower bounds.

Definition 3.47. A number α is called a *greatest lower bound* or *infimum* of A if (i) α is a lower bound of A, and (ii) If L is a lower bound of A, then $L \leq \alpha$.

Remark 3.48. Geometrically, an infimum of A lies to the left of A, but is the rightmost such point. Infima are unique (if they exist), so we are justified in writing $\inf A$. For practice, write out an alternative characterization of infima corresponding to Proposition 3.46.

Proposition 3.49. *Let A be a bounded set of real numbers, and let k be a real number.*

(i) $\sup(k + A) = k + \sup A$.

(ii) *If* $k \geq 0$, *then* $\sup(kA) = k \sup A$.

(iii) *If* $k > 0$, *then* $\sup(-kA) = -k \inf A$.

Example 3.50. If $A = [a, b]$ is an interval, then $k + A = [k + a, k + b]$. If $k \geq 0$, then $kA = [ka, kb]$ and $-kA = [-kb, -ka]$. In each case, the suprema can be read off by inspection. This is a useful way of remembering the conclusion of Proposition 3.49.

Proof. The strategy in each part is to show that the right-hand side satisfies the two conditions for a supremum in Definition 3.43.

(i) Since $k + A = \{x' \text{ in } \mathbf{R} : x' = k + x \text{ for some } x \text{ in } A\}$, a real number U is an upper bound of A if and only if $x \leq U$ for all x in A,

$$\begin{aligned} &\text{if and only if } x' = k + x \leq k + U \text{ for all } x' \text{ in } k + A, \\ &\text{if and only if } k + U \text{ is an upper bound of } k + A. \end{aligned}$$

But $\sup A$ is an upper bound of A, so $k + \sup A$ is an upper bound of $k + A$. By Definition 3.43 (ii), $\sup(k + A) \leq k + \sup A$.

Conversely, $\sup(k+A)$ is an upper bound of $k+A$, so $\sup(k+A)-k$ is an upper bound of A. By Definition 3.43 (ii), $\sup A \leq \sup(k+A)-k$, or $k + \sup A \leq \sup(k + A)$.

(ii) If $k = 0$, then $kA = \{0\}$ and the claim is obvious. Assume, therefore, that $0 < k$. The proof in this case is nothing but a mechanical modification of the preceding argument. The main idea is that multiplication or division by k preserves the sense of an inequality since $0 < k$, so a real number U is an upper bound of A if and only if kU is an upper bound of kA. Just as in the preceding argument, $k \sup A$ is an upper bound of kA, so $\sup(kA) \leq k \sup A$, and $\sup(kA)/k$ is an upper bound of A, so $k \sup A \leq \sup(kA)$.

(iii) Since $-k < 0$, multiplication or division by $-k$ *reverses* inequalities. That is, L is a *lower* bound of A if and only if $U' = -kL$ is an *upper* bound of $-kA$. Particularly, $-k \inf A$ is an upper bound of $-kA$, so $\sup(-kA) \le -k \inf A$, and $\sup(-kA)/(-k)$ is a lower bound of A, so $\sup(-kA)/(-k) \le \inf A$, or $-k \inf A \le \sup(-kA)$. ☐

The Extended Real Numbers

Definition 3.51. Let $+\infty$ and $-\infty$ denote objects that are not real numbers, and extend the ordering on the real numbers by declaring that $-\infty < x < +\infty$ for every real number x. The set $\overline{\mathbf{R}} = \mathbf{R} \cup \{-\infty, +\infty\}$ is called the *extended real number system*.

Remark 3.52. As yet we make no attempt to define addition or multiplication involving $+\infty$ or $-\infty$.

Definition 3.53. If $A \subseteq \mathbf{R}$ is arbitrary, we write $\sup A = +\infty$ if and only if A is not bounded above, and write $\inf A = -\infty$ if and only if A is not bounded below.

Remark 3.54. *Every* set A of real numbers has an infimum and a supremum in the extended real number system.

If $A = \varnothing$, then $\sup A = -\infty$ (because every extended real number is an upper bound) and $\inf A = +\infty$. The empty set is the only set whose supremum is smaller than its infimum. If $a \in A$, then

$$-\infty \le \inf A \le a \le \sup A \le +\infty.$$

Proposition 3.55. *Let A and B be non-empty sets of real numbers. If $A \subseteq B$, then* $\inf B \le \inf A \le \sup A \le \sup B$.

Proof. Every lower bound of B is *a fortiori* a lower bound of A. In particular, $\inf B \le \inf A$. Similarly, $\sup A \le \sup B$. ☐

The Archimedean Property

Theorem 3.56 (The Archimedean property)**.** *For every number x, there exists a non-negative integer n such that $x < n$.*

Remark 3.57. In words, the set $\mathbf{N}$ of natural numbers is not bounded above in $\mathbf{R}$. (Note that $\mathbf{N}$ *is* bounded above in $\overline{\mathbf{R}}$; $+\infty$ is an upper bound.)

Proof. Suppose to the contrary that there exists a real number x such that $n \leq x$ for every natural number n. This means, by definition, that $\mathbf{N}$ is bounded above in $\mathbf{R}$. By the completeness axiom, there exists a least upper bound ω in $\mathbf{R}$.

By Proposition 3.46 with $\varepsilon = 1$, the real number $\omega - 1 < \omega$ is not an upper bound of $\mathbf{N}$, so there exists a natural number n_0 such that $\omega - 1 < n_0$. But we would then have $\omega < n_0 + 1$, and $n_0 + 1 \in \mathbf{N}$ by the definition of the natural numbers, contradicting the fact that $\omega = \sup \mathbf{N}$. Contrapositively, if the axioms for the real numbers are logically consistent, then $\mathbf{N}$ is not bounded above in $\mathbf{R}$. □

Corollary 3.58 (The generalized Archimedean Property)**.** *If M is an arbitrary real number and $\varepsilon > 0$ is an arbitrary positive real number, there exists a positive integer n such that $M < n\varepsilon$.*

Remark 3.59. Metaphorically, a journey of 1000 miles (M) can be accomplished one step (ε) at a time, no matter how small the steps are.

Proof. Let $x = M/\varepsilon$. By the Archimedean property, there exists a positive integer n such that $M/\varepsilon < n$. Multiplying through by the positive number ε gives $M < n\varepsilon$. □

Corollary 3.60. *If ε is a real number and $0 < \varepsilon$, there exists a positive integer n such that $1/n < \varepsilon$. Contrapositively, if $\varepsilon \leq 1/n$ for every positive integer n, then $\varepsilon \leq 0$.*

Proof. Set $x = 1/\varepsilon$, and note that $0 < x$. By the Archimedean property, there exists an integer n such that $0 < x < n$. Theorem 2.18 (v) implies that $1/n < \varepsilon$. □

Remark 3.61. This corollary asserts, informally, that there are no infinitesimal real numbers, i.e., no positive numbers that are smaller than every reciprocal of a positive integer.

Example 3.62. The Archimedean property has noteworthy consequences for families of sets. For each positive integer n, define the open intervals $A_n = (0, n)$ and $B_n = (0, \frac{1}{n})$. The Archimedean property implies

$$\bigcup_{n=1}^{\infty} A_n = \{x \text{ in } \mathbf{R} : x > 0\}, \qquad \bigcap_{n=1}^{\infty} B_n = \varnothing.$$

In particular, a union of bounded intervals may be unbounded, and an intersection of nested, non-empty sets may be empty.

3.4 Topology

Definition 3.63. Suppose $A \subseteq \mathbf{R}$, and let $A^c = \mathbf{R} \setminus A$. For each x_0 in $\mathbf{R}$, exactly one of the following conditions holds:

(i) There exists an $\varepsilon > 0$ such that $B_\varepsilon(x_0) \subseteq A$. In this case we say x_0 is an *interior point* of A.

(ii) There exists an $\varepsilon > 0$ such that $B_\varepsilon(x_0) \subseteq A^c$. In this case we say x_0 is an *exterior point* of A.

(iii) For every $\varepsilon > 0$, the sets $B_\varepsilon(x_0) \cap A$ and $B_\varepsilon(x_0) \cap A^c$ are both non-empty. In this case we say x_0 is a *boundary point* of A.

Remark 3.64. An interior point of A is obviously an element of A, since $x_0 \in B_\varepsilon(x_0)$ regardless of ε. Similarly, an exterior point of A is never an element of A. However, a boundary point of A may lie in either A or its complement.

Example 3.65. Let $A = [a, b)$ be a half-open interval. The interior points of A comprise the open interval (a, b). There are two boundary points, a and b, one of which is an element of A and the other not. Every other real number is an exterior point.

Example 3.66. Every integer is a boundary point of $\mathbf{Z}$, and every non-integer is an exterior point of $\mathbf{Z}$.

By Theorem 3.81 below, every real number is a boundary point of $\mathbf{Q}$, the set of rational numbers.

Every real number is an interior point of $\mathbf{R}$.

The next definition distinguishes two ways the *punctured* ball $B_\varepsilon^\times(x_0)$ may be situated with respect to A.

Definition 3.67. Let A be a non-empty set of real numbers.

If $x_0 \in A$ but there exists an $\varepsilon > 0$ such that $B_\varepsilon^\times(x_0) \cap A = \varnothing$, then x_0 is called an *isolated point* of A.

If the set $B_\varepsilon^\times(x_0) \cap A$ is non-empty for every $\varepsilon > 0$, then x_0 is called a *limit point* of A (whether or not $x_0 \in A$).

Proposition 3.68. *Suppose $A \subseteq \mathbf{R}$. If x_0 is a boundary point of A, then x_0 is either an isolated point of A or a limit point of A.*

Proof. By hypothesis, $B_\varepsilon(x_0) \cap A$ is non-empty for every $\varepsilon > 0$. If $B_\varepsilon^\times(x_0) \cap A$ is empty for some $\varepsilon > 0$, then $x_0 \in A$, so x_0 is an isolated point of A. Otherwise x_0 is a limit point of A by definition. $\square$

Example 3.69. If $A = [a, b)$ is a half-open interval, then every point of the closed interval $[a, b]$ is a limit point of A, and there are no isolated points of A. The same conclusion holds for an open interval or a closed interval.

Example 3.70. Every element of $\mathbf{Z}$ is an isolated point of $\mathbf{Z}$. Similarly, if $q > 0$ is an integer, then $\frac{1}{q}\mathbf{Z}$ consists entirely of isolated points.

Definition 3.71. Let A be a set of real numbers.

We say A is an *open set* if every element of A is an interior point of A, i.e., if A contains *none* of its boundary points (see Definition 3.63).

We say A is a *closed set* if every limit point of A is an element of A, i.e., if A contains *all* of its boundary points (see Theorem 3.73).

Remark 3.72. Generally, a set contains some, but not all, of its boundary points, and is therefore neither open nor closed.

Theorem 3.73. *If $A \subseteq \mathbf{R}$, the following are equivalent:*

(i) *A is closed.*

(ii) *A contains all of its boundary points.*

(iii) *A^c is open.*

Proof. If $x_0 \notin A$, then x_0 is a boundary point of A if and only if for every $\varepsilon > 0$, $A \cap B_\varepsilon(x_0) \neq \varnothing$, if and only if for every $\varepsilon > 0$, $A \cap B_\varepsilon^\times(x_0) \neq \varnothing$, if and only if x_0 is a limit point of A.

By the preceding paragraph, A contains all its boundary points if and only if every point of A^c is not a boundary point of A, if and only if every point of A^c is not a limit point of A, if and only if A contains all its limit points, if and only if A is closed.

Since the conditions in Definition 3.63 are mutually exclusive, A contains all of its boundary points if and only if every point of A^c is an exterior point of A, a.k.a. an interior point of A^c, if and only if A^c is open. $\square$

Proposition 3.74. *An arbitrary open interval (a, b) is an open set. An arbitrary closed interval $[a, b]$ is a closed set.*

Proof. Let $a < b$ be real.

If $x \in (a, b)$, i.e., if $a < x < b$, put $\varepsilon = \min(x - a, b - x)$. Clearly $\varepsilon > 0$ and $a \leq x - \varepsilon < x < x + \varepsilon \leq b$, so $B_\varepsilon(x) \subseteq (a, b)$; that is, each element x is an interior point of (a, b).

To prove $[a, b]$ is closed, it suffices by Theorem 3.73 to prove that $\mathbf{R} \setminus [a, b] = (-\infty, a) \cup (b, \infty)$ is open. If $x < a$, set $\varepsilon = a - x > 0$ and note that $B_\varepsilon(x) \subseteq (-\infty, a)$. Similarly, if $b < x$, take $\varepsilon = x - b > 0$. $\square$

Definition 3.75. Let A be a set of real numbers. The set of interior points of A is called the *interior* of A. The set of interior and boundary points of A is called the *closure* of A.

Proposition 3.76. *A union of open sets is open, and a finite intersection of open sets is open. Precisely:*

(i) *If* $\{O_i\}_{i\in I}$ *is an arbitrary family of open subsets of* $\mathbf{R}$*, then* $\bigcup_i O_i$ *is open.*

(ii) *If* $\{O_i\}_{i=1}^n$ *is a finite family of open subsets of* $\mathbf{R}$*, then* $\bigcap_i O_i$ *is open.*

Remark 3.77. By Theorem 3.73 and De Morgan's laws for families of sets (see Exercise 1.11), an intersection of closed sets is closed, and a finite union of closed sets is closed.

Proof. (i) Let $\{O_i\}_{i\in I}$ be an arbitrary family of open sets, and let O denote the union $\bigcup_i O_i$. If $x \in O$, then $x \in O_i$ for some i. Since O_i is open, there exists an $\varepsilon > 0$ such that $B_\varepsilon(x) \subseteq O_i$. But $O_i \subseteq O$, so x is an interior point of O; since $x \in O$ was arbitrary, the union O is open.

(ii) Let $\{O_i\}_{i=1}^n$ be a finite family of open subsets of $\mathbf{R}$, and let O denote the intersection $\bigcap_i O_i$. If $x \in O$, then $x \in O_i$ for all i, so there exist positive numbers ε_i, $1 \leq i \leq n$, such that $B_{\varepsilon_i}(x) \subseteq O_i$. The minimum, $\varepsilon = \min(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n)$, is positive, and for all i we have $B_\varepsilon(x) \subseteq B_{\varepsilon_i}(x) \subseteq O_i$, so $B_\varepsilon(x) \subseteq O$. $\square$

Proposition 3.78. *The Cantor set* K *is closed, contains no interval, and contains no isolated points.*

Proof. (K is closed). Each set K_n is a finite union of closed intervals, hence is closed. Thus $K = \bigcap_n K_n$ is an intersection of closed sets, and hence is closed.

(K contains no interval). It suffices to prove K contains no *open* interval (α, β), since every interval contains an open interval. The strategy formalizes Figure 3.2. Every open interval (α, β) has positive length. However, each "approximating" set K_n is a union of intervals of length 3^{-n}, and Corollary 2.27 guarantees $3^{-n} \leq 1/(2n + 1)$.

By the Archimedean property, no positive real number is smaller than $1/(2n+1)$ for *all* n. That is, for every open interval (α, β), there exists an n such that $(\alpha, \beta) \not\subseteq K_n$. This implies $(\alpha, \beta) \not\subseteq K$. Since (α, β) was arbitrary, K contains no interval.

(K contains no isolated points). In other words, every point of K is a limit point of K: If $x_0 \in K$, then *every* deleted ball $B_\varepsilon^\times(x_0)$ contains a point of K. The strategy is similar to the preceding paragraph. If $\varepsilon > 0$ is arbitrary, there exists an n such that $3^{-n} < \varepsilon$. But $x_0 \in K$ and $K \subseteq K_n$, i.e., x_0 is an element of some component of K_n. Both endpoints of this component are closer than 3^{-n} to x_0, and at least one endpoint, say x, differs from x_0. This means that $x \in K \cap B_\varepsilon^\times(x_0)$. ☐

Density of the Rational Numbers

Definition 3.79. A set A of real numbers is *dense* (in $\mathbf{R}$) if every real number is a boundary point of A, i.e., if for every real number x and every $\varepsilon > 0$, there exists a number x_0 in A such that $|x - x_0| < \varepsilon$.

Remark 3.80. If A is dense in $\mathbf{R}$, then every real number can be approximated arbitrarily closely by elements of A.

Theorem 3.81. *Let $x < y$ be distinct real numbers. There exists a non-zero rational number $r = m/n$ such that $x < r < y$.*

Remark 3.82. In words, the set $\mathbf{Q}$ of rational numbers is dense in $\mathbf{R}$: If $x \in \mathbf{R}$ and if $\varepsilon > 0$, there exists a rational number $r = p/q$ such that $x < r < y = x + \varepsilon$.

Proof. Assume first that $0 \leq x < y$. The real number $\varepsilon = y - x$ is positive because $x < y$. By Corollary 3.60, there exists a positive integer n such that $1/n < \varepsilon$. It suffices to prove that some integer multiple $r = m/n$ lies between x and y. Geometrically this is plausible: By taking steps of size $1/n$ across the interval between x and y, we must step in the interior at least once.

Formally, consider the set $S = \{p \text{ in } \mathbf{Z}^+ : y \leq p/n\}$. By the generalized Archimedean property, there is a positive integer p such that $y < p/n$; that is, the set S is non-empty. Now, a non-empty set of positive integers has a smallest element, say $m + 1$. By definition of the set S, we have $m/n < y \leq (m + 1)/n$. To complete the proof, it suffices to prove $x < m/n$. But

$$y - \frac{m}{n} \leq \frac{m+1}{n} - \frac{m}{n} = \frac{1}{n} < y - x.$$

Rearranging the inequality gives $x < m/n$. We have proven that if $0 \leq x < y$, then there exists a non-zero rational number $r = m/n$ such that $x < r < y$.

If instead $x < y \leq 0$, then $0 \leq -y < -x$. By the preceding argument, there exists a rational number $r \neq 0$ such that $-y < r < -x$, so the non-zero rational number $-r$ satisfies $x < -r < y$.

Finally, if neither alternative holds, then $x < 0 < y$, and by the first part of the proof there is a rational number $x < 0 < r < y$. □

Remark 3.83. Theorem 3.81 says that between any two real numbers can be found a rational number. Moreover, between any two real numbers can be found an irrational number. For example, if $x < y$, then by Theorem 3.81 there exists a non-zero rational number r such that $x/\sqrt{2} < r < y/\sqrt{2}$, i.e., such that $x < r\sqrt{2} < y$. But $r\sqrt{2}$ is irrational by Proposition 2.50.

It is tempting to conclude that rational and irrational numbers "alternate" along the number line, as if by painting rational numbers red and irrational numbers blue, the number line would consist of alternating red and blue points. Unfortunately, this picture is utterly incorrect. Distinct real numbers are never adjacent to each other, but instead are endpoints of an interval containing infinitely many rational and irrational numbers.

Exercises

Exercise 3.1. Show that:

(a) If $x \approx \mathsf{A}(1)$, then $x^2 \approx \mathsf{A}(|x|)$ and $\big(1 + \mathsf{A}(|x|)\big)^2 \approx 1 + 3\mathsf{A}(|x|)$.

(b) If $x \approx \mathsf{A}(0.75)$, then $0.25 < 1 + x < 1.75$ and $1/(1 + x) \approx \mathsf{A}(4)$.

(c) If $x \approx \mathsf{A}(1)$, what can be said about $1/(1 + x)$?

Exercise 3.2. If $x \approx 1.56 + \mathsf{A}(0.005)$ and $y \approx 2.13 + \mathsf{A}(0.005)$, calculate $x + y$, and show that $xy \approx 3.588 + \mathsf{A}(0.02) \approx 3.6 + \mathsf{A}(0.04)$.

Exercise 3.3. In each part, let x and h be real numbers, $h \neq 0$.

(a) Show $(x + h)^2 - x^2 \approx 2xh + \mathsf{A}(h^2)$.

(b) Show $(x + h)^3 - x^3 \approx 3x^2h + (3|x| + |h|)\mathsf{A}(h^2)$.

(c) Find an analogous formula for $(x + h)^4 - x^4$.

(d) Use part (a) to show

$$\frac{(x+h)^2 - x^2}{h} \approx 2x + \mathsf{A}(|h|).$$

Find corresponding formulas using parts (b) and (c).

Exercise 3.4. If $r > 0$ and x_0 are real, let $B_r(x_0)$ denote the deleted neighborhood of radius r about x_0. Prove that
(a) $B_r^\times(0) = rB_1^\times(0)$. (b) $B_r^\times(x_0) = x_0 + B_r^\times(0)$.

Exercise 3.5. Let $a < b$ be real numbers. Find real numbers $r > 0$ and x_0 such that
(a) $[a, b] = x_0 + r[0, 1]$. (b) $[a, b] = x_0 + r[-1, 1]$.

Exercise 3.6. Let $A = \{\frac{1}{n} : n \in \mathbf{Z}^+\}$. Make a sketch of A, and determine which real numbers (if any) are interior points of A, exterior points of A, boundary points of A, isolated points of A, limit points of A. Is A closed?

Exercise 3.7. For each positive real number r, let $O_r = B_r^\times(0)$ be the deleted neighborhood of radius r about 0. Determine $\bigcap_r O_r$ with justification.

Exercise 3.8. Let A be a bounded set of real numbers, and let k be a real number. Formulate and prove a version of Proposition 3.49 for infima.

Exercise 3.9. Let x_0 be a real number. Find an infinite family $\{O_n\}$ of open sets whose intersection is $\{x_0\}$. (This shows that Proposition 3.76 (ii) does not extend to arbitrary families of open sets.)

Exercise 3.10. For $n > 1$, let $A_n = [\frac{1}{n}, \frac{n}{n+1}]$. Express the union $\bigcup_n A_n$ as an interval, with proof.

Exercise 3.11. For each positive integer n, let $I_n = [a_n, b_n]$ be a *closed*, bounded interval of real numbers. Prove that if the intervals are nested inward, i.e., if $I_{n+1} \subseteq I_n$, then the infinite intersection $\bigcap_n I_n$ is nonempty. (Compare Example 3.62, in which the sets B_n are not closed.) Hint: Consider the sets $A = \{a_n\}$ and $B = \{b_n\}$. Show that A is bounded above, B is bounded below, and $\sup A \le \inf B$; conclude that $[\sup A, \inf B]$ is contained in $\bigcap_n I_n$.

Exercise 3.12. Let O be a non-empty open set of real numbers. This exercise outlines a proof that O is a union of open intervals.

(a) Suppose $x \in O$, and put

$$a_x = \inf\{t : (t, x) \subseteq O\}, \qquad b_x = \sup\{t : (x, t) \subseteq O\}.$$

Prove that $a_x < x < b_x$ (note strict inequalities).

(b) In the notation of part (a), prove that $(a_x, b_x) \subseteq O$, and that no strictly large open interval is contained in O. (An interval of the form (a_x, b_x) is called a *component* of O.)

(c) Prove that O is the union over x in O of the components (a_x, b_x).

Exercise 3.13. Let $O = \{x \text{ in } \mathbf{R} : x \neq 0, x \neq \pm 1/n \text{ for all } n \geq 1\}$. Sketch the set O, show O is open, and find the components of O.

Exercise 3.14. Let A_1, A_2, A_3, …, be non-empty sets of real numbers, each consisting entirely of isolated points.

(a) Prove that $A_1 \cup A_2$ consists entirely of isolated points.
Hint: Suppose $x_0 \in A_1 \cup A_2$. If $B^\times_{\varepsilon_1}(x_0) \cap A_1 = \varnothing$ and $B^\times_{\varepsilon_2}(x_0) \cap A_2 = \varnothing$, let $\varepsilon = \min(\varepsilon_1, \varepsilon_2)$.

(b) Prove that $A_1 \cup A_2 \cup \cdots \cup A_n$ consists entirely of isolated points. In particular, $\mathbf{Q}_N$ consists entirely of isolated points.

(c) Show by example that the infinite union $\bigcup_n A_n$ may consist entirely of isolated points, or may be dense in $\mathbf{R}$.

Chapter 4

Sequences

Definition 4.1. Let k_0 be a non-negative integer. A *real sequence* is a collection of ordered pairs (k, a_k) in which $k \geq k_0$ is an integer and a_k is a real number. We call a_k the *kth term* of the sequence.

Remark 4.2. The "first coordinate" k imposes an ordering on the terms a_k. A sequence is often denoted $(a_k)_{k=k_0}^{\infty}$, or simply by (a_k), in order to emphasize its interpretation as an infinite ordered list of real numbers. The ordering of the terms is crucial; a sequence must be carefully distinguished from its *set of terms* $A = \{a_k : k \geq k_0\} \subseteq \mathbf{R}$.

By convention, $k_0 = 0$ unless the contrary is explicitly stated. In this case, using the terminology of Chapter 5, a real sequence is a "mapping" $\mathbf{a} : \mathbf{N} \to \mathbf{R}$, and $a_k = \mathbf{a}(k)$ is the "value" of $\mathbf{a}$ at k.

Example 4.3. The formulas $a_k = 1/(k+1)$ and $b_k = (-1)^k$ define real sequences. The respective sets of terms are $A = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\}$, and the finite set $B = \{1, -1\}$.

Example 4.4. For each real number x, the formula $a_k = x^k$ defines a real sequence.

Example 4.5. The recursive specification

$$a_0 = 2, \quad a_{k+1} = \frac{1}{2}\left(a_k + \frac{2}{a_k}\right)$$

defines a real sequence. The next three terms are $a_1 = 3/2$, $a_2 = 17/12$, and $a_3 = 577/408$.

Example 4.6. The recursive rule $a_{k+1} = a_k + \bigl[1/(k+1)\bigr]$, with $a_0 = 0$, defines a sequence whose nth term ($n \geq 1$) may be written

$$a_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} = \sum_{k=1}^{n} \frac{1}{k}.$$

4.1 Convergence

Definition 4.7. Let (a_k) be a real sequence, and let $a_\infty \in \mathbf{R}$. We say (a_k) *converges* to a_∞, and write $(a_k) \to a_\infty$, if the following condition holds:

> For every $\varepsilon > 0$, there exists a natural number N such that
> if $k \geq N$, then $|a_k - a_\infty| < \varepsilon$.

Remark 4.8. The definition may alternatively be expressed:

> For every $\varepsilon > 0$, there exists an index N such that if $k \geq N$,
> then $a_k \approx a_\infty + \mathsf{A}(\varepsilon)$.

In words, the terms a_k can be made "as close as we like" to a_∞ (i.e., within an error of $\varepsilon > 0$, no matter how small ε is chosen) by "going sufficiently far out into the sequence" (i.e., considering only terms a_k with $k \geq N$). Loosely, "the a_k eventually get arbitrarily close to a_∞".

A crucial aspect of the condition is that ε is chosen *before* N.

Proposition 4.9. *If (a_k) is a real sequence that converges to a_∞ and to a'_∞, then $a_\infty = a'_\infty$.*

Proof. Let $\varepsilon > 0$ be arbitrary. Since $(a_k) \to a_\infty$, there exists an index N_1 such that if $k \geq N_1$, then $|a_k - a_\infty| < \varepsilon/2$. Similarly, since $(a_k) \to a'_\infty$, there exists an index N'_1 such that if $k \geq N'_1$, then $|a_k - a'_\infty| < \varepsilon/2$.

Let $N = \max(N_1, N'_1)$. Since $N \geq N_1$ and $N \geq N'_1$, the triangle inequality implies

$$\begin{aligned} |a'_\infty - a_\infty| &= |(a'_\infty - a_N) + (a_N - a_\infty)| \\ &\leq |a'_\infty - a_N| + |a_N - a_\infty| < (\varepsilon/2) + (\varepsilon/2) = \varepsilon. \end{aligned}$$

But $\varepsilon > 0$ was arbitrary, which means $a'_\infty - a_\infty = 0$. ☐

Remark 4.10. Proposition 4.9 says that a real sequence converges to at most one number. If $(a_k) \to a_\infty$, we call a_∞ the *limit* of (a_k), and write $a_\infty = \lim\limits_{k\to\infty} a_k$.

Remark 4.11. Informally, we say "a_k *approaches* a_∞ as $k \to \infty$". Indeed, the notation a_∞ is meant to suggest "setting $k = \infty$ in the limit". Logically, however, this is not what convergence means. The terms a_k of a sequence (a_k) are merely individual real numbers, which do not "approach" anything. It is the *sequence* (i.e., the *ordered list of terms*) that converges (or fails to converge) to a limit.

The ε-N Game, and Examples

Convergence of a sequence may be understood as an adversarial game. A sequence (a_k) and a putative limit a_∞ are specified in advance. The first player chooses a positive "tolerance" ε, which defines a target, the interval $B_\varepsilon(a_\infty) = (a_\infty - \varepsilon, a_\infty + \varepsilon)$.

The second player now tries to "hit the target", i.e., to ensure that $|a_k - a_\infty| < \varepsilon$, solely by taking k to be sufficiently large. A "successful response" to the first player's "challenge" is a positive integer N such that if $k \geq N$, then $|a_k - a_\infty| < \varepsilon$.

If Player N is able to respond successfully to a particular ε, they "win the round". Otherwise Player ε wins the round.

To say $(a_k) \to a_\infty$ means that Player N *has a winning strategy against a perfect opponent*: No matter how "skillful" Player ε is (i.e., no matter how small $\varepsilon > 0$ is chosen), Player N can always issue a successful response.

Success in analysis is largely a matter of learning to play the ε-N game and its variants (to be studied later). When you read proofs, you may notice that seemingly magical choices of N are made. To make these choices, the author imagined an arbitrary $\varepsilon > 0$ was given, and formulated a strategy for choosing a "winning" index N. The proof itself is merely a written demonstration that Player N wins.

Remark 4.12. Yet another interpretation of convergence of a sequence has the procedural appeal of a solitaire game. Imagine an infinite deck of playing cards in which the kth card has the term a_k of the sequence written on its face. The cards are placed in a row in order, face up. Now an $\varepsilon > 0$ is specified, and for each k, if $|a_k - a_\infty| \geq \varepsilon$, the kth card is turned face down. At issue is whether all the cards past some point remain face up, i.e., whether *only finitely many cards must be turned.* If this is the case *no matter how $\varepsilon > 0$ is specified*, we say $(a_k) \to a_\infty$.

Example 4.13. Let $c \in \mathbf{R}$. The sequence $a_k = c$ is called a *constant sequence.* A constant sequence obviously converges to $a_\infty = c$: For every $\varepsilon > 0$ and for every k, we have $|a_k - a_\infty| = |c - c| = 0 < \varepsilon$. That is, Player N *cannot lose against a perfect opponent* when playing with a constant sequence!

Example 4.14. The sequence $a_k = 1/(k+1)$ converges to 0. Before giving a proof, we'll play a few rounds of the ε-N game.

If $\varepsilon = 100$, Player N *cannot lose*, and in particular may take $N = 0$: Indeed, $|a_k - 0| = 1/(k+1) \leq 1 < \varepsilon$ regardless of k. Note carefully that this fact by itself does not prove $(a_k) \to 0$; Player N must be able to win against an *arbitrary* $\varepsilon > 0$.

If $\varepsilon = 0.01 = 1/100$, player N may take $N = 100$: If $k \geq 100$, then $|a_k - 0| = 1/(k+1) \leq 1/101 < 1/100 = \varepsilon$.

If $\varepsilon = 1/\sqrt{200}$ (assuming such a real number exists), Player N's goal is to find an N such that $1/(N+1) < \varepsilon = 1/\sqrt{200}$, or after rearranging, $200 < (N+1)^2$. The "best" (i.e., smallest) choice, $N = 14$, is easy to find in this example, but there is no harm in taking, for example, $N = 200$, which is surely sufficient.

To show $(a_k) \to 0$, it suffices to construct a winning strategy for Player N. Let $\varepsilon > 0$ be arbitrary. By Corollary 3.60 of the Archimedean property, there exists a positive integer N such that $1/N < \varepsilon$. This N is our response. If $k \geq N$, then $|a_k - 0| = 1/(k+1) < 1/N < \varepsilon$.

Remark 4.15. If an index N "wins" against some challenge $\varepsilon > 0$, then *every larger integer $N' \geq N$ also wins*, because $k \geq N'$ implies $k \geq N$. It is not necessary (or desirable) to pick the smallest winning N.

Correspondingly, making ε smaller makes the target smaller, which makes the condition $a_k \approx a_\infty + \mathsf{A}(\varepsilon)$ is "harder to meet", and generally forces N to be larger.

Remark 4.16. The standard idiom for picking a *single* N satisfying *finitely many conditions* is to pick multiple Ns, each satisfying one condition, then let our response be the largest of our choices. This was done already in the proof of Proposition 4.9.

Example 4.17. The sequence $a_k = (-1)^k$ has terms that are alternately 1 and -1; precisely, the "even" terms $a_{2\ell}$ are all 1 and the "odd" terms $a_{2\ell+1}$ are all -1. We will show that (a_k) has no limit.

That is, for every real number a_∞, the statement "$(a_k) \to a_\infty$" is false. To prove this, we fix a putative limit a_∞ arbitrarily, then take the side of Player ε and look for a winning strategy:

> No matter what N is, $k = 2N \geq N$ is even, and $k = 2N+1 \geq N$ is odd. We are therefore assured that $|a_k - a_\infty|$ takes both values $|1 - a_\infty|$ and $|-1 - a_\infty| = |1 + a_\infty|$ for some $k \geq N$. In order to win, Player N must make *both* of these quantities smaller than ε. But in that event, the triangle inequality would imply $2 \leq |1 - a_\infty| + |1 + a_\infty| < \varepsilon + \varepsilon = 2\varepsilon$. If this inequality is not satisfied, Player N loses.

Having reasoned thusly, Player ε chooses any ε with $0 < \varepsilon \leq 1$. For definiteness, take $\varepsilon = 1$. By the preceding reasoning, there does not exist a positive integer N such that if $k \geq N$, then $|a_k - a_\infty| < \varepsilon = 1$; if such an N existed, we would have

$$2 \leq |1 - a_\infty| + |1 + a_\infty| = |a_{2N} - a_\infty| + |a_{2N+1} + a_\infty| < \varepsilon + \varepsilon = 2,$$

which is false. Since "$(a_k) \to a_\infty$" is false for every real number a_∞, the sequence $(a_k) = \big((-1)^k\big)$ has no limit.

Theorem 4.18. *Let x be a real number satisfying $-1 < x \leq 1$. The sequence $a_k = x^k$ converges; the limit is 1 if $x = 1$, and is 0 otherwise.*

Proof. If $x = 1$, the sequence (a_k) is constant (since $a_k = 1^k = 1$ for all k), and therefore converges to 1 as noted above. Similarly, if $x = 0$, then $a_k = 0$ for all k.

For the remainder of the proof, assume $0 < |x| < 1$, and set $a_\infty = 0$. By Corollary 2.27, if we write $|x| = 1/(1 + u)$, then $|x^k| \leq 1/(1 + ku)$ for all $k \geq 0$.

Fix $\varepsilon > 0$ arbitrarily. By the generalized Archimedean property, there exists a natural number N such that $1/(Nu) < \varepsilon$. If $k \geq N$, then

$$|x^k - a_\infty| = |x^k| \leq \frac{1}{1 + ku} < \frac{1}{ku} \leq \frac{1}{Nu} < \varepsilon.$$

By definition, this means $(x^k) \to a_\infty = 0$. $\square$

Example 4.19. For $x = 1/2$ or $x = -4/5$, say, the conclusion of Theorem 4.18 is not intuitively surprising. However, for a number such as $x = 0.99999999999999999999 = 1 - 10^{-20}$, a "fairly large" exponent n may be needed to make the power x^n "small".

Remark 4.20. The sequences in Examples 4.5 and 4.6 cannot be handled so naively from the definition. We turn next to theoretical criteria that help resolve these examples.

Boundedness

Definition 4.21. A real sequence (a_k) is *bounded above* if its set of terms is bounded above, i.e., if there exists a real number M such that $a_k \leq M$ for all k. Any particular M is called an *upper bound* for the sequence.

Similarly, we say (a_k) is *bounded below* if there exists a real number m such that $m \leq a_k$ for all k.

We say the real sequence (a_k) is *bounded* if (a_k) is bounded above and bounded below, namely (Proposition 3.36), if there exists a positive real number M such that $-M \leq a_k \leq M$, i.e., $|a_k| \leq M$, for all k.

Remark 4.22. Geometrically, a sequence is bounded above if every term lies to the left of some fixed real number, and is bounded below if every term lies to the right of some (other) number.

Proposition 4.23. *If (a_k) is a convergent real sequence with limit a_∞, then*

(i) *(a_k) is bounded.*

(ii) *The sequence $\big(|a_k|\big)$ converges to $|a_\infty|$.*

Proof. By hypothesis, for every $\varepsilon > 0$, there exists an N such that if $k \geq N$, then $|a_k - a_\infty| < \varepsilon$.

(i) Taking $\varepsilon = 1$, there exists an N such that $|a_k - a_\infty| < 1$ for $k \geq N$. Let $M = \max(|a_0|, |a_1|, \dots, |a_{N-1}|, |a_\infty| + 1)$. It suffices to show $|a_k| \leq M$ for all k.

If $0 \leq k < N$, then $|a_k| \leq M$ by construction. If $k \geq N$, the triangle inequality implies

$$|a_k| \leq |a_\infty| + |a_k - a_\infty| < |a_\infty| + 1 \leq M.$$

In summary, $|a_k| \leq M$ for all k.

(ii) Let $\varepsilon > 0$, and choose N such that if $k \geq N$, then $|a_k - a_\infty| < \varepsilon$. By the reverse triangle inequality, $\big||a_k| - |a_\infty|\big| \leq |a_k - a_\infty| < \varepsilon$ if $k \geq N$. Since $\varepsilon > 0$ was arbitrary, $\big(|a_k|\big) \to |a_\infty|$. □

Monotone Sequences

Definition 4.24. Let (a_k) be a real sequence.

If $a_k \leq a_{k+1}$ for all k, namely if $a_k \leq a_{k'}$ whenever $k \leq k'$, we say (a_k) is *non-decreasing*.

If $a_{k+1} \leq a_k$ for all k, namely if $a_{k'} \leq a_k$ whenever $k \leq k'$, we say (a_k) is *non-increasing.*

A sequence that is either non-decreasing or non-increasing is said to be *monotone.*

A slightly more general condition is useful in practice, and does not require much extra work to accommodate.

Definition 4.25. A real sequence (a_k) is *eventually non-decreasing* if there exists an index N_0 such that $a_N \leq a_k$ whenever $N_0 \leq N \leq k$.

Example 4.26. The real sequence defined by $a_k = k^2/(k^2 - 500)$ is eventually non-decreasing, since

$$\frac{k^2}{k^2 - 500} = \frac{k^2 - 500 + 500}{k^2 - 500} = 1 - \frac{500}{k^2 - 500},$$

and if $k^2 > 500$, i.e., if $k \geq 23 = N_0$, the denominator increases with k.

Example 4.27. The real sequences defined by $a_k = (-1)^k k$ or by $a_k = (-1)^k/k$ are *not* eventually non-decreasing.

For practice, give definitions of *eventually non-increasing* and *eventually monotone* real sequences, and modify of the statement and proof of Theorem 4.28 below to handle these types of sequence.

Theorem 4.28. *If (a_k) is an eventually non-decreasing sequence, then*

(i) *(a_k) is bounded below.*

(ii) *(a_k) converges if and only if it is bounded above.*

Proof. By hypothesis, there exists an integer N_0 such that $a_N \leq a_k$ whenever $N_0 \leq N \leq k$. Particularly, $a_{N_0} \leq a_k$ if $N_0 \leq k$.

(i) The real number $m = \min(a_0, a_1, \dots, a_{N_0})$ is a lower bound for (a_k): By construction, $m \leq a_k$ for $0 \leq k \leq N_0$. On the other hand, if $N_0 \leq k$, then $m \leq a_{N_0} \leq a_k$ as noted above.

(ii) (Convergent implies bounded above). By Proposition 4.23, a convergent sequence is bounded, hence bounded above.

(Bounded above implies convergent). Suppose there exists a real number M such that $a_k \leq M$ for all k. The set $A = \{a_k : N_0 \leq k\}$ is non-empty and bounded above by M, so $a_\infty = \sup A$ exists by the completeness axiom. It suffices to prove $(a_k) \to a_\infty$.

Let $\varepsilon > 0$ be arbitrary. The number $a_\infty - \varepsilon < a_\infty$ is not an upper bound of A, so there exists an integer $N \geq N_0$ such that $a_\infty - \varepsilon < a_N$. If $k \geq N$, then $a_N \leq a_k \leq \sup A$, so

$$a_\infty - \varepsilon < a_N \leq a_k \leq a_\infty < a_\infty + \varepsilon.$$

That is, if $k \geq N$, then $a_k \approx a_\infty + \mathsf{A}(\varepsilon)$. Since $\varepsilon > 0$ was arbitrary, we have shown $(a_k) \to a_\infty$. $\square$

Example 4.29. The sequence $a_k = 1/(k+1)$ is non-increasing, and is bounded below since every term is positive. Consequently, (a_k) converges to $a_\infty = \inf\{a_k\}$. Of course, we have already seen that this sequence converges to 0; here we have deduced a less specific conclusion using a more general theorem.

Example 4.30. Consider the sequence (a_k) in Example 4.5. By mathematical induction, each term is positive and satisfies $a_k^2 - 2 > 0$. Further, the sequence is non-increasing since

$$a_k - a_{k+1} = a_k - \frac{1}{2}\left(a_k + \frac{2}{a_k}\right) = \frac{1}{2}\left(a_k - \frac{2}{a_k}\right) = \frac{a_k^2 - 2}{2a_k} > 0.$$

Consequently, $\lim a_k$ exists. Note that we have shown that this sequence converges without explicitly exhibiting the limit.

Algebraic Properties of Limits

Theorem 4.31. *Let (a_k) and (b_k) be real sequences converging to a_∞ and b_∞ respectively.*

(i) *$(a_k + b_k)$ converges to $a_\infty + b_\infty$.*

(ii) *$(a_k b_k)$ converges to $a_\infty b_\infty$.*

(iii) *If $b_k \neq 0$ for all k and if $b_\infty \neq 0$, then (a_k/b_k) converges to a_∞/b_∞.*

Proof. (i) Let $\varepsilon > 0$ be arbitrary. By hypothesis, there exists an integer N_1 such that if $k \geq N_1$, then $a_k \approx a_\infty + \mathsf{A}(\varepsilon/2)$. Similarly, there exists an integer N_2 such that if $k \geq N_2$, then $b_k \approx b_\infty + \mathsf{A}(\varepsilon/2)$. Let $N = \max(N_1, N_2)$.

By Theorem 3.18, if $k \geq N$, then $a_k + b_k \approx a_\infty + b_\infty + \mathsf{A}(\varepsilon)$. Since $\varepsilon > 0$ was arbitrary, $(a_k + b_k) \to a_\infty + b_\infty$.

(ii) By Proposition 4.23, there exist positive real numbers L and M such that $1+|a_k| \le L$ and $1+|b_k| \le M$ for all k. Let ε be an arbitrary real number with $0 < \varepsilon < 1$, and put $r = \varepsilon/2M < \varepsilon$ and $s = \varepsilon/2L$.

Since $(a_k) \to a_\infty$, there exists an integer N_1 such that if $k \ge N_1$, then $a_k \approx a_\infty + \mathsf{A}(r)$, and since $(b_k) \to b_\infty$, there exists an integer N_2 such that if $k \ge N_2$, then $b_k \approx b_\infty + \mathsf{A}(s)$.

Let $N = \max(N_1, N_2)$, and note that $|a_\infty| < |a_N| + \varepsilon < L$. If $k \ge N$, Theorem 3.18 (ii) implies

$$\begin{aligned} a_k b_k &\approx a_\infty b_\infty + A\big(|a_\infty| s + Mr\big) \\ &\approx a_\infty b_\infty + A\big(L \cdot \varepsilon/2L + M \cdot \varepsilon/2M\big) \\ &\approx a_\infty b_\infty + \mathsf{A}(\varepsilon). \end{aligned}$$

Since $\varepsilon > 0$ was arbitrarily small, $(a_k b_k) \to a_\infty b_\infty$.

(iii) We first prove that $(1/b_k) \to 1/b_\infty$. Fix $\varepsilon > 0$. With the goal of applying Proposition 3.20, put $r = \frac{1}{2}\min(|b_\infty|,\ \varepsilon \cdot |b_\infty|^2)$. Note that $r > 0$ because $|b_\infty| > 0$, and $2r/|b_\infty|^2 \le \varepsilon$ by construction.

Because $(b_k) \to b_\infty$, there exists an integer N such that if $k \ge N$, then $b_k \approx b_\infty + \mathsf{A}(r)$. By Proposition 3.20 with $x = b_k$ and $x_0 = b_\infty$,

$$\frac{1}{b_k} \approx \frac{1}{b_\infty} + \mathsf{A}\left(\frac{2r}{|b_\infty|^2}\right) \approx \frac{1}{b_\infty} + \mathsf{A}(\varepsilon).$$

(iii) now follows immediately from (ii) by writing $\dfrac{a_k}{b_k} = a_k \cdot \dfrac{1}{b_k}$. □

Remark 4.32. The preceding proof uses our "computational" A notation. To give the "standard" proofs, first translate $a_k \approx a_\infty + \mathsf{A}(\varepsilon/2)$ into $|a_k - a_\infty| < \varepsilon/2$, etc., throughout.

For (i), the translation is essentially complete. For (ii), choose positive real numbers M, L, and an integer N as above, then invoke algebraic identities and the triangle inequality; if $k \ge N$, then

$$\begin{aligned} |a_k b_k - a_\infty b_\infty| &= |a_k b_k - a_\infty b_k + a_\infty b_k - a_\infty b_\infty| \\ &\le |a_k b_k - a_\infty b_k| + |a_\infty b_k - a_\infty b_\infty| \\ &= |a_k - a_\infty|\,|b_k| + |a_\infty|\,|b_k - b_\infty| \\ &< \frac{\varepsilon}{2M} \cdot M + \frac{\varepsilon}{2L} \cdot L < \varepsilon. \end{aligned}$$

For (iii), pick $r > 0$ as above, and write

$$\left|\frac{1}{b_k} - \frac{1}{b_\infty}\right| = \frac{|b_\infty - b_k|}{|b_k|\,|b_\infty|} < \frac{2r}{|b_\infty|^2} \le \varepsilon.$$

In short, the "standard" proof focuses on the difference between a term of a sequence and the limit. The A notation proof writes the limit as the sum of a term and an error, and focuses on bounding the error.

Example 4.33. Let m be a positive integer. The sequence $(a_k)_{k=1}^\infty$ defined by $a_k = k^{-m} = 1/k^m$ converges to 0. When $m = 1$, this assertion is proven in Example 4.14. For larger m, Theorem 4.31 (ii) establishes the inductive step.

Example 4.34. In practice, Theorem 4.31 is often invoked without formally defining a sequence. The theorem grants license to move a limit into or out of an arithmetic expression:

$$\begin{aligned}
\lim_{k\to\infty} \frac{k-1}{k+1} &= \lim_{k\to\infty} \frac{1-(1/k)}{1+(1/k)} = \frac{1-\lim_k(1/k)}{1+\lim_k(1/k)} = \frac{1}{1} = 1;\\
\lim_{k\to\infty} \frac{2k}{k^2+1} &= \lim_{k\to\infty} \frac{(2k)/k^2}{1+(1/k^2)} = \frac{2\lim_k(1/k)}{1+\lim_k(1/k^2)} = \frac{0}{1} = 0;\\
\lim_{k\to\infty} \left(\frac{k+1}{k}\right)^m &= \left(\lim_{k\to\infty} \frac{k+1}{k}\right)^m = (1)^m = 1; \quad \text{etc.}
\end{aligned}$$

In the third example, the integer exponent $m \geq 0$ is arbitrary, but "fixed", i.e., independent of k. Bringing the limit inside the parentheses implicitly involves an inductive argument together with Theorem 4.31.

Note carefully that Theorem 4.31 does *not* imply

$$\lim_{k\to\infty} \left(\frac{k+1}{k}\right)^k = 1.$$

As we will see in Chapter 10, this limit is not 1, but e, an irrational number that pervades pure and applied mathematics.

Limits and Inequalities

Proposition 4.35. *Let (a_k) and (b_k) be convergent real sequences, with respective limits a_∞ and b_∞.*

(i) *If $0 \leq a_k$ for all but finitely many k, then $0 \leq a_\infty$.*

(ii) *If $a_k \leq b_k$ for all but finitely many k, then $a_\infty \leq b_\infty$.*

(iii) *If $c \leq a_k \leq d$ for all but finitely many k, then $c \leq a_\infty \leq d$.*

Remark 4.36. In words, "non-strict inequality is preserved in the limit". Note that *strict* inequality is not generally preserved in the limit: We have $1/k > 0$ for all $k \geq 1$, but $\lim_k 1/k = 0$.

Proof. (i) We prove the contrapositive: If $a_\infty < 0$, then $a_k < 0$ for infinitely many k.

Put $\varepsilon = -a_\infty/2$, so $\varepsilon > 0$ by hypothesis. Since $(a_k) \to a_\infty$, there exists an N such that if $k \geq N$, then $|a_k - a_\infty| < \varepsilon$. But $|a_k - a_\infty| < \varepsilon$ if and only if $a_\infty/2 < a_k - a_\infty < -a_\infty/2$, and this implies $a_k < a_\infty/2 < 0$. That is, if $k \geq N$, then $a_k < 0$.

(ii) Define $c_k = b_k - a_k$. By hypothesis, $0 \leq c_k$ for all but finitely many k. By Theorem 4.31, (c_k) converges to $b_\infty - a_\infty$. Part (i) implies $0 \leq b_\infty - a_\infty$, i.e., $a_\infty \leq b_\infty$.

(iii) This follows immediately from (ii). $\square$

Divergence to Infinity

The definition of sequential convergence makes no sense if $a_\infty = \infty$ or $a_\infty = -\infty$. The symbols ∞ and $-\infty$ are not real numbers, so formal inequalities such as $|a_k - \infty| < \varepsilon$ have no meaning.

Nonetheless, it is useful to be able to study sequences that "approach" ∞ or $-\infty$. Such sequences diverge (i.e., do not have a real limit), but they still enjoy some special properties of convergent sequences.

Definition 4.37. Let (a_k) be a real sequence. We say (a_k) *diverges to* ∞, denoted $(a_k) \to \infty$, if the following condition holds:

> For every real number M, there exists a natural number N such that if $k \geq N$, then $a_k > M$.

We say (a_k) *diverges to* $-\infty$, denoted $(a_k) \to -\infty$, if:

> For every real number M, there exists a natural number N such that if $k \geq N$, then $a_k < M$.

Example 4.38. The sequence $a_k = k$ diverges to infinity: By the Archimedean property, if $M \in \mathbf{R}$, there exists a natural number N such that $N > M$. If $k \geq N$, then $a_k = k \geq N > M$.

Example 4.39. Fix $x > 1$. The sequence $a_k = x^k$ (see Example 4.4) diverges to infinity. To see this, note that $x = 1 + u$ for some $u > 0$.

By Theorem 2.26, $1 + ku \leq x^k = a_k$ for all $k \geq 0$. Fix $M \in \mathbf{R}$, and use the generalized Archimedean property to choose a natural number N such that $Nu > M$. If $k \geq N$, then

$$M < Nu < 1 + Nu \leq 1 + ku \leq x^k = a_k.$$

Since M was an arbitrary real number, $(x^k) \to \infty$.

Remark 4.40. If (a_k) is a real sequence that is eventually non-decreasing, then either (a_k) is bounded above (hence convergent to a finite limit), or not bounded above (hence divergent to ∞).

Analogous remarks hold for a real sequence that is eventually non-increasing. Consequently, an eventually monotone sequence *always* has an extended real limit.

Theorem 4.41. *Assume $(a_k) \to \infty$ and $(b_k) \to b_\infty$ with b_∞ real. Then*

(i) *$(a_k + b_k) \to \infty$.*

(ii) *If $b_\infty > 0$, then $(a_k b_k) \to \infty$. If $b_\infty < 0$, then $(a_k b_k) \to -\infty$.*

(iii) *If $a_k \neq 0$ for all k, then $(b_k/a_k) \to 0$.*

Proof. (i) Since a convergent sequence is bounded (Proposition 4.23) and $(b_k) \to b_\infty$, there exists a real number $B > 0$ such that $|b_k| \leq B$ for all k. Let M be arbitrary. Since $(a_k) \to \infty$, there exists an N such that if $k \geq N$, then $a_k > M + B$, which implies

$$a_k + b_k \geq a_k - |b_k| > (M + B) - B = M.$$

Since M was arbitrary, $(a_k + b_k) \to \infty$.

(ii) Assume $b_\infty > 0$. Taking $\varepsilon = b_\infty/2 > 0$, there exists an N_1 such that if $k \geq N_1$, then $|b_k - b_\infty| < \varepsilon = b_\infty/2$. Rearranging gives $b_\infty/2 < b_k$ for $k \geq N_1$.

Fix M arbitrarily. Since $(a_k) \to \infty$, there exists an $N \geq N_1$ such that if $k \geq N$, then $a_k > 2M/b_\infty$. But this implies

$$a_k b_k > (2M/b_\infty) \cdot (b_\infty/2) = M,$$

and since M was arbitrary, $(a_k b_k) \to \infty$. To handle the case $b_\infty < 0$, multiply appropriately by -1 in the preceding proof.

(iii) Suppose $a_k \neq 0$ for all k, so the quotient sequence (b_k/a_k) is defined. As in (i), let $B > 0$ be a bound for $|b_k|$. Fix $\varepsilon > 0$ arbitrarily. Since $(a_k) \to \infty$, there is an N such that if $k \geq N$, then $a_k > B/\varepsilon$, which implies $b_k/a_k < B(\varepsilon/B) = \varepsilon$. Since $\varepsilon > 0$ was arbitrary, $(b_k/a_k) \to 0$. □

Remark 4.42. Theorem 4.41 may be interpreted as assigning values to certain arithmetic expressions containing infinity: If $L > 0$ is real, then

$$\infty \pm L = \infty, \qquad \infty \cdot (\pm L) = \pm\infty, \qquad \pm L/\infty = 0.$$

Easy modifications of the preceding arguments establish that

$$\infty + \infty = \infty, \qquad -\infty - \infty = -\infty, \qquad \pm\infty \cdot \infty = \pm\infty.$$

However, ∞ and $-\infty$ are not real numbers, and the preceding "equations" must be understood as theorems about limits. By contrast, the following expressions are *undefined*, in the sense that their value depends on the sequences used to approximate them:

$$\infty - \infty, \qquad 0 \cdot (\pm\infty), \qquad \infty/\infty, \qquad 0/0.$$

Consequently, care is required when manipulating algebraic expressions involving ∞. For example, it is true (in the "limited" sense above) that $1 + \infty = \infty$, but not legitimate to subtract ∞, "deducing" that $1 = 0$.

4.2 Subsequences

Definition 4.43. An *index sequence* is a strictly increasing sequence ν of natural numbers, i.e., $\nu(k) \in \mathbf{N}$ and $\nu(k) < \nu(k + 1)$ for all k.

Two useful properties are easily verified by mathematical induction.

Lemma 4.44. *Let ν be an index sequence.*

(i) *If $k < k'$, then $\nu(k) < \nu(k')$.*

(ii) *$k \leq \nu(k)$ for all k, and if the inequality is strict for some k_0, then the inequality is strict for all $k \geq k_0$.*

Definition 4.45. Let $\mathbf{a}$ be a real sequence and ν an index sequence. The sequence $\mathbf{b}$ defined by $b_k = a_{\nu(k)}$ is called a *subsequence* of $\mathbf{a}$.

Remark 4.46. In words, a subsequence of (a_n) is a sequence obtained by selecting an infinite number of terms $a_{\nu(1)}$, $a_{\nu(2)}$, ..., $a_{\nu(k)}$, ..., *in their original ordering*, i.e., subject to $\nu(1) < \nu(2) < \cdots < \nu(k) < \ldots$.

Example 4.47. The subsequence (a_{2k}), for which we take $\nu(k) = 2k$, consists of the *even terms* of (a_n). The subsequence (a_{2k+1}), taking $\nu(k) = 2k + 1$, consists of the *odd terms* of (a_n).

Example 4.48. Let $(a_n)_{n=0}^\infty$ be a real sequence. For each N in $\mathbf{N}$, the subsequence $(a_n)_{n=N}^\infty = (a_{N+k})_{k=0}^\infty$ is a *tail* of (a_n), obtained by discarding the terms $a_0, \dots, a_{N-1}$. Here $\nu(k) = N + k$.

Remark 4.49. Convergence of a sequence is determined solely by convergence of an arbitrary tail; intuitively, prepending or omitting finitely many terms cannot change the convergence or divergence of a sequence.

Remark 4.50. If some tail of a sequence has a property X, we say the original sequence is "eventually X". Terms such as "eventually non-decreasing" have already been introduced. Similarly, we might say a sequence is "eventually positive", "eventually no larger than 1 in absolute value", or "eventually constant".

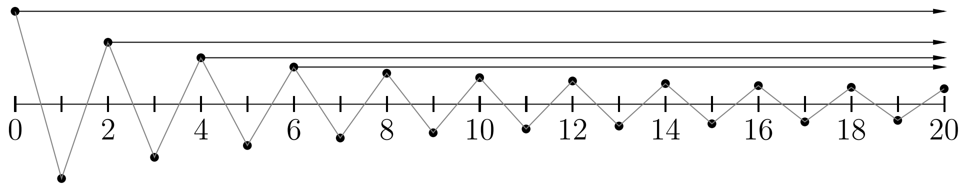
Conditions such as "eventually bounded" or "eventually convergent" are syntactically "legal", but carry no meaning, because a tail cannot be bounded, unbounded, convergent, or divergent unless the original sequence already possesses the same property.

Proposition 4.51. *Let (a_n) be a real sequence that converges to a_∞. If $(b_k) = (a_{\nu(k)})$ is an arbitrary subsequence, then $(b_k) \to a_\infty$.*

Proof. Fix $\varepsilon > 0$ arbitrarily, and pick an index N such that if $n \geq N$, then $|a_n - a_\infty| < \varepsilon$. If $k \geq N$, then $\nu(k) \geq N$ by Lemma 4.44, so $|b_k - a_\infty| = |a_{\nu(k)} - a_\infty| < \varepsilon$. □

Definition 4.52. Let $\mathbf{a}$ be a real sequence. An index n is a *peak point* of $\mathbf{a}$ if $a_m \leq a_n$ for all $m > n$.

Remark 4.53. Intuitively, a peak point is a location n from which, standing at height a_n and looking to the right, you can see all the way to infinity.



Example 4.54. If $a_n = (-1)^n 4/(4+n)$, then every even natural number is a peak point. (Arrows indicate unobstructed lines of sight.)

Example 4.55. A sequence $\mathbf{a}$ is non-increasing if and only if every natural number is a peak point of $\mathbf{a}$.

If $\mathbf{a}$ is non-decreasing, then $\mathbf{a}$ has no peak points.

If $\mathbf{a}$ is unbounded above, then $\mathbf{a}$ has no peak points.

Theorem 4.56. *Every real sequence has a monotone subsequence.*

Proof. We consider two cases: The sequence $\mathbf{a}$ has infinitely many peak points, or only finitely many. In each case, we construct a monotone subsequence recursively.

(Infinitely many peak points). Let $\nu(1)$ be a peak point. Now let $m \geq 1$, and assume inductively that we have constructed peak points $\nu(1) < \nu(2) < \cdots < \nu(m)$. Since $\mathbf{a}$ has infinitely many peak points, there exists a peak point $\nu(m+1) > \nu(m)$. The subsequence $(a_{\nu(k)})$ is non-increasing: By definition of a peak point, $a_{\nu(k+1)} \leq a_{\nu(k)}$ for all k.

(Finitely many peak points). Since there are only finitely many peak points, there exists an integer $\nu(1)$ that is greater than every peak point. Now let $m \geq 1$, and assume inductively that we have constructed indices $\nu(1) < \nu(2) < \cdots < \nu(m)$ such that $a_{\nu(1)} < a_{\nu(2)} < \cdots < a_{\nu(m)}$. Since $\nu(m)$ is not a peak point, there exists an index $\nu(m+1) > \nu(m)$ such that $a_{\nu(m)} < a_{\nu(m+1)}$. This completes the recursive step.

The subsequence $(a_{\nu(k)})$ is increasing by construction. ☐

Corollary 4.57 (The Bolzano-Weierstrass theorem)**.** *Every bounded real sequence has a convergent subsequence.*

Proof. Let $\mathbf{a}$ be a bounded sequence. By the theorem, there exists a monotone subsequence $(a_{\nu(k)})$. But a bounded, monotone sequence converges by Theorem 4.28. ☐

4.3 Cauchy Sequences

In order to prove from the definition that a real sequence (a_k) converges, the correct "candidate" limit a_∞ must be known. In practice, unfortunately, the limit is not known. The "Cauchy criterion" reformulates the definition of convergence in a way that refers only to the terms of the sequence itself.

Definition 4.58. A real sequence $\mathbf{a}$ is a *Cauchy sequence* if the following condition holds:

> For every $\varepsilon > 0$, there exists an index N such that if k and $k' \geq N$, then $|a_{k'} - a_k| < \varepsilon$.

Remark 4.59. Exchanging k and k' has no effect on $|a_{k'} - a_k|$, so (as dictated by convenience) we may assume without loss of generality that $k' > k$, or that $k' < k$.

Instead of specifying that $k' > k$, we often write $k' = k + m$ with m a positive integer. The "Cauchy predicate" becomes "If $k \geq N$ and $m > 0$, then $|a_{k+m} - a_k| < \varepsilon$".

Theorem 4.60. *Let (a_k) be a real sequence. Then (a_k) converges to some real number a_∞ if and only if (a_k) is a Cauchy sequence.*

Proof. (Convergent implies Cauchy). Assume $(a_k) \to a_\infty$, and fix $\varepsilon > 0$ arbitrarily. There exists a natural number N such that if $k \geq N$, then $|a_k - a_\infty| < \varepsilon/2$. Consequently, if k and $k' \geq N$, then

$$|a_{k'} - a_k| \leq |a_{k'} - a_\infty| + |a_\infty - a_k| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, (a_k) is Cauchy.

(Cauchy implies convergent). The proof proceeds as follows. We first prove that every Cauchy sequence is bounded. By the Bolzano-Weierstrass theorem, a Cauchy sequence has a convergent subsequence. Finally, we prove that a Cauchy sequence having a convergent subsequence is itself convergent to the same limit.

Taking $\varepsilon = 1$ in the Cauchy criterion, there is an index N such that if k and $k' \geq N$, then $|a_{k'} - a_k| < 1$. In particular, $|a_k - a_N| < 1$ for all $k \geq N$. Let $M = \max(|a_0|, |a_1|, \dots, |a_{N-1}|, |a_N| + 1)$. Just as in the proof of Proposition 4.23 (ii), it follows that $|a_k| \leq M$ for all k.

Since the Cauchy sequence (a_k) is bounded, the Bolzano-Weierstrass theorem guarantees there is a subsequence $(a_{\nu(k)})$ converging to some real number a_∞. To complete the proof, it suffices to show $(a_k) \to a_\infty$.

Fix $\varepsilon > 0$ arbitrarily, and pick an index N_1 such that if $k \geq N_1$, then $|a_{\nu(k)} - a_\infty| < \varepsilon/2$. Now use the Cauchy criterion to pick $N \geq N_1$ such that if k and $k' \geq N$, then $|a_k - a_{k'}| < \varepsilon/2$.

Since $\nu(N) \geq N$, we have $|a_{\nu(N)} - a_\infty| < \varepsilon/2$, and $|a_k - a_{\nu(N)}| < \varepsilon/2$ for all $k \geq N$. By the triangle inequality, if $k \geq N$, then

$$|a_k - a_\infty| \leq |a_k - a_{\nu(N)}| + |a_{\nu(N)} - a_\infty| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, $(a_k) \to a_\infty$. □

4.4 Infinite Series

Definition 4.61. Let $(a_k)_{k=0}^{\infty}$ be a real sequence. We define the sequence (s_n) of *partial sums* as follows:

$$s_n = \sum_{k=0}^{n} a_k = a_0 + a_1 + a_2 + \cdots + a_n.$$

Precisely, the partial sums are defined recursively:

$$s_0 = a_0, \qquad s_{n+1} = s_n + a_{n+1} \quad \text{for } n \geq 0.$$

The sequence (a_k) is *summable* if the sequence of partial sums converges to a finite limit s. In this event, we write

$$s = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \sum_{k=0}^{n} a_k = \sum_{k=0}^{\infty} a_k.$$

The expression on the right is called the *infinite series* with *terms* or *summands* a_k, and is said to *converge* to s.

If the sequence of partial sums does not converge, we say the infinite series $\sum_k a_k$ *diverges.*

Remark 4.62. Think of a real sequence as an infinite list of credits (non-negative terms) and debits (negative terms). The partial sums are the "running totals" of the terms *taken in a specified order*. The sum of a series, if it exists, is the net value in the limit, when "all the terms have been added" in their specified order.

Remark 4.63. Since $s_{n+1} - s_n = a_{n+1}$, if $a_k \geq 0$ for all k, the sequence of partial sums is non-decreasing. More generally, if only finitely many a_k are negative, the sequence of partial sums is eventually non-decreasing. Similar remarks hold if at most finitely many summands are positive.

A few general properties are useful to record before we turn to examples.

Theorem 4.64. *Let (a_k) and (b_k) be summable real sequences, and assume $c \in \mathbf{R}$. The sequences $(a_k + b_k)$ and (ca_k) are summable, and*

$$\sum_{k=0}^{\infty} (a_k + b_k) = \sum_{k=0}^{\infty} a_k + \sum_{k=0}^{\infty} b_k, \qquad \sum_{k=0}^{\infty} (ca_k) = c \sum_{k=0}^{\infty} a_k.$$

Proof. If (s_n) and (t_n) denote the respective sequences of partial sums of (a_k) and (b_k), then $(s_n + t_n)$ and (cs_n) are the respective partial sums of $(a_k + b_k)$ and (ca_k). The theorem follows immediately from Theorem 4.31. □

Proposition 4.65. *Let (a_k) be a real sequence. For all natural numbers N and m, we have:*

$$s_{N+m} - s_N = \sum_{k=N+1}^{N+m} a_k, \quad \textit{and therefore} \quad |s_{N+m} - s_N| \leq \sum_{k=N+1}^{N+m} |a_k|.$$

Theorem 4.66. *A real sequence (a_k) is summable if and only if the sequence of partial sums is Cauchy. In particular, if (a_k) is summable, then $(a_k) \to 0$.*

Proof. The first assertion is immediate from Theorem 4.60. For the second, fix $\varepsilon > 0$, and choose an index N such that if k and $k' \geq N$, then $|s_{k'} - s_k| < \varepsilon$. In particular, if $k \geq N$, then $|a_{k+1}| = |s_{k+1} - s_k| < \varepsilon$. This means $(a_k) \to 0$. □

Example 4.67. Let $a \neq 0$ and r be real numbers. The infinite series

$$\sum_{k=0}^{\infty} ar^k = a + ar + ar^2 + ar^3 + \dots$$

is called the *geometric series* with first term a and ratio r.

If $|r| \geq 1$, then $|ar^k| = |a|\,|r|^k$ does not converge to 0, see Example 4.39, so the geometric series diverges.

Suppose $-1 < r < 1$. The key to analyzing the geometric series is the algebraic observation that multiplying a partial sum by r "shifts" the summands:

$$\begin{aligned} s_n &= \sum_{k=0}^{n} ar^k = a + ar + ar^2 + ar^3 + \dots + ar^n, \\ rs_n &= \sum_{k=1}^{n+1} ar^k = \phantom{a+{}} ar + ar^2 + ar^3 + \dots + ar^n + ar^{n+1}. \end{aligned}$$

Subtracting the second line from the first, $(1 - r)s_n = a - ar^{n+1}$. Since $r \neq 1$, this equation can be solved for s_n:

$$s_n = \sum_{k=0}^{n} ar^k = a + ar + ar^2 + ar^3 + \dots + ar^n = \frac{a(1 - r^{n+1})}{1 - r},$$

the *finite geometric series formula.* (This formula is correct for all real r other than 1.)

By Example 4.4, $(r^n) \to 0$ since $-1 < r < 1$. Theorem 4.31 implies

$$\sum_{k=0}^{\infty} ar^k = \lim_{n\to\infty} \sum_{k=0}^{n} ar^k = \lim_{n\to\infty} \frac{a(1-r^{n+1})}{1-r} = \frac{a(1-\lim_n r^{n+1})}{1-r} = \frac{a}{1-r}.$$

Example 4.68. The so-called *harmonic series*

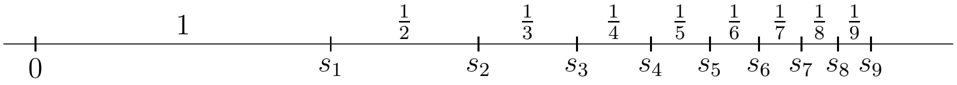
$$\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} + \dots$$

introduced in Example 4.6 has terms that decrease to 0. Nonetheless, the harmonic series diverges. To prove this, it suffices to show the partial sums are unbounded.

Each of the first two terms is at least $1/2$. The next two terms, $1/3$ and $1/4$, are each no smaller than $1/4$, so their sum is greater than $2 \cdot 1/4 = 1/2$.

The next four terms, $1/5$, $1/6$, $1/7$, and $1/8$, are each at least $1/8$, so their sum is greater than $4 \cdot 1/8 = 1/2$.

Similarly, the next eight terms, $1/9, 1/10, \dots, 1/16$, sum to at least $8 \cdot 1/16 = 1/2$, the sixteen terms after that sum to at least $1/2$, and so on *ad infinitum.*



In more detail, if $m \geq 1$ is an integer, then

$$\sum_{k=2^{m-1}+1}^{2^m} \frac{1}{k} \geq \sum_{k=2^{m-1}+1}^{2^m} \frac{1}{2^m} = \frac{2^{m-1}}{2^m} = \frac{1}{2}.$$

Consequently,

$$\sum_{k=1}^{2^n} \frac{1}{k} = 1 + \sum_{m=1}^{n} \sum_{k=2^{m-1}+1}^{2^m} \frac{1}{k} \geq 1 + \sum_{m=1}^{n} \frac{1}{2} = 1 + \frac{n}{2},$$

which is unbounded by the Archimedean property.

Tests for Summability

No single, easily-applied test determines whether a general infinite series is convergent or divergent. Instead, we develop techniques to handle special classes of series.

Lemma 4.69. *Let (a_k) and (b_k) be real sequences, and let (s_n) and (t_n) denote the sequences of partial sums of (a_k) and (b_k) respectively. If (b_k) is summable, and if there exists a natural number N_0 such that*

$$|s_{m'} - s_m| \leq |t_{m'} - t_m| \quad \text{whenever } m' > m \geq N_0,$$

then (a_k) is summable.

Proof. If (b_k) is summable, i.e., if the sequence (t_n) of partial sums converges, then (t_n) is Cauchy. Thus, for every $\varepsilon > 0$, there exists an index $N \geq N_0$ such that if $m' > m \geq N$, then $|t_{m'} - t_m| < \varepsilon$. By the hypotheses of the lemma, if $m' > m \geq N$, then $|s_{m'} - s_m| < \varepsilon$ as well. This implies the sequence (s_n) is Cauchy, hence convergent by Theorem 4.60. □

Theorem 4.70 (The comparison test)**.** *Let (a_k) and (b_k) be* non-negative *real sequences, and assume $a_k \leq b_k$ for all but at most finitely many k. If (b_k) is summable, then (a_k) is summable. Contrapositively, if (a_k) is not summable, then (b_k) is not summable.*

Proof. Let (s_n) and (t_n) denote the sequences of partial sums of (a_k) and (b_k) respectively. By hypothesis, there exists an index N_0 such that $a_k \leq b_k$ for all $k \geq N_0$. If $n \geq N_0$ and $m > 0$, then

$$|s_{n+m} - s_n| = \sum_{k=n+1}^{n+m} a_k \leq \sum_{k=n+1}^{n+m} b_k = |t_{n+m} - t_n|.$$

The theorem follows from Lemma 4.69. □

Absolute Summability

Theorem 4.71. *Let (a_k) be a real sequence. If the sequence $(|a_k|)$ of absolute values is summable, then (a_k) is summable, and*

$$\left|\sum_{k=0}^{\infty} a_k\right| \leq \sum_{k=0}^{\infty} |a_k|.$$

Proof. Let (s_n) and (t_n) denote the sequences of partial sums of (a_k) and $(|a_k|)$ respectively. By the triangle inequality,

$$|s_{n+m} - s_n| = \left|\sum_{k=n+1}^{n+m} a_k\right| \leq \sum_{k=n+1}^{n+m} |a_k| = |t_{n+m} - t_n|$$

for all natural numbers n and m. If (t_n) converges, then (s_n) converges by Lemma 4.69. Since

$$\left|\sum_{k=0}^{n} a_k\right| \leq \sum_{k=0}^{n} |a_k| \quad \text{for all } n \geq 0$$

the inequality in the theorem holds by Proposition 4.35. □

Definition 4.72. A real sequence (a_k) is *absolutely summable* if the sequence $(|a_k|)$ is summable.

If (a_k) is summable but $(|a_k|)$ is not, then (a_k) is *conditionally summable.*

Remark 4.73. Alternatively, an infinite series $\sum_k a_k$ is *absolutely convergent* if $\sum_k |a_k|$ is convergent, and is *conditionally convergent* if $\sum_k a_k$ converges but $\sum_k |a_k|$ diverges.

Definition 4.74. Let (a_k) be a real sequence. The sequences (a_k^+) and (a_k^-) defined by

$$a_k^+ = \max(a_k, 0) = \frac{|a_k| + a_k}{2}, \qquad a_k^- = -\min(a_k, 0) = \frac{|a_k| - a_k}{2},$$

are called the sequence of *positive terms* of (a_k) and the sequence of *negative terms* of (a_k), respectively.

Example 4.75. If $a_k = (-1/2)^k$ for $k \geq 0$, then

$k =$	0	1	2	3	4	5	…
$a_k =$	1	$-1/2$	$1/4$	$-1/8$	$1/16$	$-1/32$	…
$\lvert a_k\rvert =$	1	$1/2$	$1/4$	$1/8$	$1/16$	$1/32$	…
$a_k^+ =$	1	0	$1/4$	0	$1/16$	0	…
$a_k^- =$	0	$1/2$	0	$1/8$	0	$1/32$	…

Remark 4.76. Each sequence $(a_k^\pm)$ is non-negative, so each is summable if and only if its sequence of partial sums is bounded. Further,

$$a_k = a_k^+ - a_k^-, \qquad |a_k| = a_k^+ + a_k^-,$$

and $0 \leq a_k^\pm \leq |a_k|$ for all k.

Proposition 4.77. *Let (a_k) be a real sequence.*

(i) *(a_k) is absolutely summable if and only if both sequences (a_k^+) and (a_k^-) are summable.*

(ii) *If (a_k) is conditionally summable, then both sequences (a_k^+) and (a_k^-) are non-summable.*

Proof. (i) If (a_k) is absolutely summable, i.e., if $(|a_k|)$ is summable, then each sequence $(a_k^\pm)$ is summable by comparison with $(|a_k|)$.

Conversely, if the sequences (a_k^+) and (a_k^-) are both summable, then $(|a_k|) = (a_k^+ + a_k^-)$ is summable by Theorem 4.64.

(ii) By hypothesis, (a_k) is summable but $(|a_k|)$ is not. If either of $(a_k^\pm)$ were summable, then $(|a_k|)$ would be as well by Theorem 4.64, since $|a_k| = 2a_k^\pm \mp a_k$. Contrapositively, $(a_k^\pm)$ is not summable. ☐

The Ratio Test

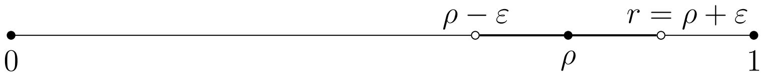
Theorem 4.78 (The ratio test)**.** *Let (a_k) be a real sequence. If the limiting ratio*

$$\rho = \lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right|$$

exists, and if $\rho < 1$, then $\sum_k a_k$ is absolutely convergent.

Proof. Let $r = (1+\rho)/2$, so that $\rho < r < 1$, and put $\varepsilon = r - \rho$, see diagram. Since $|a_{k+1}/a_k| \to \rho$, there exists an index N such that if $k \geq N$, then

$$\left|\frac{a_{k+1}}{a_k}\right| < \rho + \varepsilon = r < 1.$$



Rearranging, $|a_{k+1}| < |a_k| r$ for all $k \geq N$. By induction on m,

$$|a_{N+m}| < |a_N| r^m \quad \text{for all } m > 0.$$

Consequently, we have

$$\sum_{k=N+1}^{\infty} |a_k| = \sum_{m=1}^{\infty} |a_{N+m}| \leq |a_N| \sum_{m=1}^{\infty} r^m.$$

This upper bound is a convergent geometric series, so $\sum_k |a_k|$ converges by comparison. ☐

Corollary 4.79. *Let r be a real number with $|r| < 1$. For each positive integer m, the series $\sum_k k^m r^k$ is absolutely convergent. Consequently, the sequence $(k^m r^k)$ converges to 0, and in particular is bounded.*

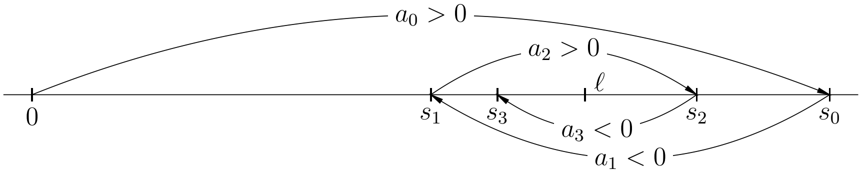
Proof. Setting $a_k = k^m r^k$, we have

$$\lim_{k\to\infty} \left| \frac{(k+1)^m r^{k+1}}{k^m r^k} \right| = \lim_{k\to\infty} \big(1 + (1/k)\big)^m |r| = |r| < 1,$$

so $\sum_k k^m r^k$ is absolutely convergent by the ratio test. The remaining assertions are immediate. □

Alternating Series

Definition 4.80. If (a_k) is a sequence of positive terms, an infinite series $\pm \sum_k (-1)^k a_k$ is said to be *alternating.*



Theorem 4.81 (The alternating series test)**.** *If (a_k) is a* non-increasing *positive real sequence, and if $(a_k) \to 0$, then the alternating series $\sum_k (-1)^k a_k$ converges, and*

$$\left| \sum_{k=n+1}^{\infty} (-1)^k a_k \right| \le |a_{n+1}|.$$

Proof. Assume for definiteness that the first summand, say a_0, is positive, and let

$$s_n = \sum_{k=0}^{n} (-1)^k a_k.$$

If n is an arbitrary even index, then $s_{n+2} = s_n - a_{n+1} + a_{n+2} \le s_n$ since $a_{n+2} \le a_{n+1}$. With the same n, $s_{n+3} = s_{n+1} + a_{n+2} - a_{n+3} \ge s_{n+1}$. Writing $n = 2m$ and combining these inequalities,

$$s_{2m+1} \le s_{2m+3} \le s_{2m+2} \le s_{2m}$$

for all $m \geq 0$. Consequently, the even partial sums form a non-increasing sequence that is bounded below by every odd partial sum, and is therefore convergent to a real number $\ell^+ = \inf_m s_{2m}$. Similarly, the odd partial sums form a non-decreasing sequence that is bounded above by every even partial sum, and is therefore convergent to a real number $\ell^- = \sup_m s_{2m+1}$. Since $(a_k) \to 0$ and

$$\ell^+ - \ell^- \leq s_{2m} - s_{2m+1} = a_{2m+1}$$

for all $m \geq 0$, we have $\ell^+ = \ell^-$, i.e., the sequence of partial sums converges to some real number ℓ, and

$$\left| \sum_{k=n+1}^{\infty} (-1)^k a_k \right| = |\ell - s_n| \leq |a_{n+1}|. \qquad \square$$

Exercises

Exercise 4.1. Consider the real sequence defined by $a_k = k^{(-1)^k}$. Show that (a_k) has a subsequence converging to 0 and a subsequence diverging to ∞.

Exercise 4.2. Construct a sequence $\mathbf{a}$ of positive real numbers such that $\mathbf{a} \to 0$ but $\mathbf{a}$ is not eventually non-decreasing.

Exercise 4.3. Let ℓ_1, ℓ_2, $\dots$, ℓ_m be arbitrary distinct real numbers. Prove there exists a real sequence $\mathbf{a}$ such that for each $i = 1, \dots, m$, some subsequence of $\mathbf{a}$ converges to ℓ_i.

Exercise 4.4. Let (a_k) be a sequence of *positive* real numbers, and let $b_k = 1/a_k$. Prove that if $(a_k) \to 0$, then $(b_k) \to \infty$.

Exercise 4.5. Let (a_k) be an integer sequence, i.e., a_k is an integer for all k. Prove that (a_k) converges if and only if (a_k) is eventually constant.

Exercise 4.6. Let (a_k) and (b_k) be real sequences. Prove that if (b_k) is bounded and $(a_k) \to 0$, then $(a_k b_k) \to 0$.

Exercise 4.7. (The squeeze theorem) Suppose (a_k), (b_k), and (c_k) are real sequences, and assume there exists an N_0 such that if $k \geq N_0$, then $a_k \leq c_k \leq b_k$. Prove that if (a_k) and (b_k) converge to the same limit L, then $\lim_k c_k$ exists is equal to L.

Exercise 4.8. Let (a_k) be an absolutely summable sequence. Prove the sequence (a_k^2) is absolutely summable.

Exercise 4.9. If (a_k) and (b_k) are real sequences, we may define a new sequence (c_k) by "shuffling" the two given sequences, i.e., $c_{2k} = a_k$ and $c_{2k+1} = b_k$ for $k \geq 0$.

Write out the first six terms of (c_k). Prove that (c_k) converges if and only if (a_k) and (b_k) converge to the same limit.

Exercise 4.10. Let (a_k) be a real sequence and a_∞ a real number. Consider the following conditions:

(i) For every $\varepsilon > 0$, there exists an N such that if $k \geq N$ then $|a_k - a_\infty| < \varepsilon$.

(ii) There exists an N such that for every $\varepsilon > 0$, if $k \geq N$ then $|a_k - a_\infty| < \varepsilon$.

Are these conditions logically equivalent? If so, give a proof. If not, find a "familiar" condition equivalent to (ii), and give an example of a sequence satisfying (ii) but not (i).

Exercise 4.11. Let (a_k) be a real sequence and a_∞ a real number. Consider the following conditions:

(i) There exists an $\varepsilon > 0$ such that for every integer $N \geq 0$, if $k \geq N$ then $|a_k - a_\infty| < \varepsilon$.

(ii) For every integer $N \geq 0$, there exists an $\varepsilon > 0$ such that if $k \geq N$ then $|a_k - a_\infty| < \varepsilon$.

Are these conditions logically equivalent? If so, give a proof. If not, find a "familiar" condition equivalent to (ii), and give an example of a sequence satisfying (ii) but not (i).

Exercise 4.12. Let $\mathbf{a}$ be a sequence of *positive* real numbers, and assume $\mathbf{a} \to 0$.

(a) Prove that for every natural number m, there exists a natural number $n > m$ such that $a_n < a_m$.

(b) Prove that $\mathbf{a}$ has a strictly decreasing subsequence. Suggestion: Use part (a) to construct a decreasing subsequence inductively.

Exercise 4.13. Let $\mathbf{a}$ be a real sequence that is not bounded above. Prove there exists a strictly increasing subsequence.

Exercise 4.14. Let $\mathbf{a}$ be a sequence that is not eventually constant.

(a) Prove that if $\mathbf{a}$ is non-increasing, then $\mathbf{a}$ has a strictly decreasing subsequence.

(b) Use the proof of Theorem 4.56 to prove $\mathbf{a}$ has a strictly monotone subsequence.

Exercise 4.15. Let $\mathbf{a}$ be a non-increasing sequence of positive real numbers, and assume $\mathbf{a} \to 0$. Prove there exists a positive, non-increasing sequence $\mathbf{b}$ such that $\mathbf{b} \to 0$ but $(b_k/a_k) \to \infty$.

Exercise 4.16. Let $(a_k)_{k=1}^{\infty}$ be a "digit sequence", i.e., a sequence all of whose terms are integers between 0 and 9 inclusive.

(a) Prove that the series

$$\sum_{k=1}^{\infty} \frac{a_k}{10^k} = \frac{a_1}{10} + \frac{a_2}{100} + \frac{a_3}{1000} + \cdots = 0.a_1a_2a_3\ldots$$

converges to a real limit between 0 and 1.
Suggestion: The geometric series formula may be useful in obtaining an upper bound.

(b) Prove that the digit sequences $(5, 0, 0, \ldots)$ and $(4, 9, 9, 9, \ldots)$ give rise to the same sum in part (a).

(c) Under what conditions do two distinct digit sequences define the same sum in part (a)?
Suggestions: Let (a_k) and (b_k) be distinct digit sequences defining the same real number, and assume (i) $a_k = b_k$ for $1 \leq k < n$, and (ii) $a_n < b_n$. How much smaller than b_n can a_n be? What can you say about all subsequent digits of each sequence?

Exercise 4.17. Let $(a_k)_{k=1}^{\infty}$ be a sequence all of whose terms are either 0 or 2. Prove that the series

$$\sum_{k=1}^{\infty} \frac{a_k}{3^k} = \frac{a_1}{3} + \frac{a_2}{3^2} + \frac{a_3}{3^3} + \ldots$$

converges to an element of the Cantor set K. Conversely, show that every element of K can be expressed (in exactly one way) in this form.
Suggestion: Consider how successive "digits" of x are related to the location of x with respect to the approximating sets K_n.

Chapter 5

Functions

Definition 5.1. Let X and Y be non-empty sets. A *function* or *mapping* $f : X \to Y$ (read "f from X to Y") is a set f of ordered pairs (x, y) with the property that for each x in X, there exists exactly one y in Y such that $(x, y) \in f$.

If $(x, y) \in f$, we write $y = f(x)$ and call y the *value of f at x*. The set X is the *domain* of f; its elements are the "inputs" of f. The set Y is the *target* of f; its elements are "potential outputs" of f.

Remark 5.2. Informally, a function $f : X \to Y$ is a "rule" assigning a unique element y in Y to each element x in X. In calculus, the set f is called the *graph* of f.

Unlike the situation in calculus, a function is not merely a formula. For example, changing the domain by removing a single point defines a *different function*.

Example 5.3. A real sequence is a mapping $\mathbf{a} : \mathbf{N} \to \mathbf{R}$. A subsequence may be viewed as the result of "shrinking" the domain from $\mathbf{N}$ to a proper subset $\{\nu(k) : k \in \mathbf{N}\} = \{\nu(1) < \nu(2) < \cdots < \nu(k) < \dots\}$.

5.1 Examples of Functions

Polynomials and Rational Functions

Example 5.4. Let X be a set of real numbers. The function $\iota : X \to X$ defined by $\iota(x) = x$ for all x in X is called the *identity function* on X.

Example 5.5. Let X be a set of real numbers. For each real number c, there is a unique *constant function* $c : X \to \mathbf{R}$ defined by $c(x) = c$ for all x in X.

Definition 5.6. Let f and g be real-valued functions with domain X. The functions $f \pm g$ and fg with domain X are defined by the formulas

$$(f \pm g)(x) = f(x) \pm g(x), \qquad (fg)(x) = f(x) \cdot g(x)$$

for each x in X. These functions are called the *pointwise* sum, difference, and product of f and g.

If $Z = \{x \text{ in } X : g(x) = 0\}$ is the *zero set* of g on X, there is a pointwise quotient f/g defined by $(f/g)(x) = f(x)/g(x)$ for all x in the set $X \setminus Z$, the *natural domain* of f/g.

Example 5.7. Let a_0, a_1, a_2, …, a_n be real numbers. The formula $f(x) = a_0 + a_1x$ defines a *linear function* $f : \mathbf{R} \to \mathbf{R}$.

The formula $f(x) = a_0 + a_1x + a_2x^2$ defines a *quadratic function* $f : \mathbf{R} \to \mathbf{R}$.

Generally, the formula

$$\begin{aligned} p(x) &= a_0 + a_1x + a_2x^2 + \cdots + a_nx^n \\ &= a_0 + x\big(a_1 + x\big(a_2 + \cdots + x(a_n)\dots\big)\big) \end{aligned}$$

defines a *polynomial function*. If $a_n \neq 0$, we say p has *degree n*.

The numbers a_0, a_1, a_2, …, a_n are the *coefficients* of p. The *leading term* is the highest degree term having non-zero coefficient.

Remark 5.8. Polynomial functions are the most general functions obtained from constant functions and the identity function ι on $\mathbf{R}$ using only the operations of addition and multiplication. The second expression above for $p(x)$, called *Horner's form*, entails at most n multiplications and n additions.

Example 5.9. Let

$$\begin{aligned} p(x) &= a_0 + a_1x + a_2x^2 + \cdots + a_nx^n, \\ q(x) &= b_0 + b_1x + b_2x^2 + \cdots + b_mx^m \end{aligned}$$

be polynomials, and let Z be the zero set of q on $\mathbf{R}$. The formula

$$f(x) = \frac{p(x)}{q(x)} = \frac{a_0 + a_1x + a_2x^2 + \cdots + a_nx^n}{b_0 + b_1x + b_2x^2 + \cdots + b_mx^m}$$

defines a *rational function* $f : \mathbf{R} \setminus Z \to \mathbf{R}$.

Example 5.10. The formulas

$$f(x) = \frac{x}{1 - x^2}, \qquad g(x) = \frac{x}{x^2 + 1}$$

define rational functions. The natural domain of f is the set $\mathbf{R} \setminus \{\pm 1\}$. The natural domain of g is $\mathbf{R}$.

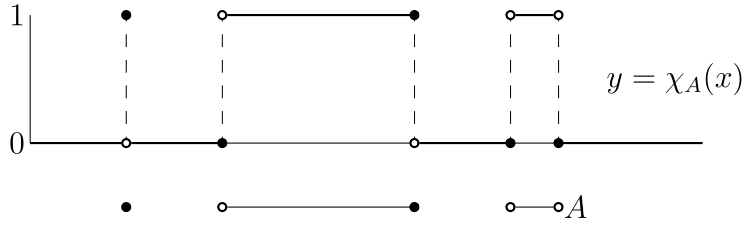
Patching

In practice, a function may be defined by partitioning the domain and using different formulas on the various subsets.

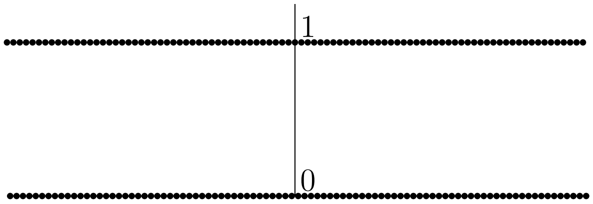
Example 5.11. Assume $A \subseteq \mathbf{R}$. The *characteristic function* of A is the function $\chi_A : \mathbf{R} \to \mathbf{R}$ defined by

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases}$$

In words, χ_A inquires of each real number x, "Are you an element of A?" and returns 1 (yes) or 0 (no) accordingly.



The characteristic function of $\mathbf{Q}$, the set of rational numbers, is known as the *Dirichlet function.*



Example 5.12. Define $f : [0, \infty) \to \mathbf{R}$ by

$$f(x) = \begin{cases} x & \text{if } 0 \le x \le 1, \\ 1/x & \text{if } 1 < x. \end{cases}$$

A function of this type is said to be *defined piecewise*: The domain is divided into finitely many intervals, and an algebraic formula is given for each subinterval.

Example 5.13. Let $[a, b]$ be a closed interval of real numbers. A function f defined on $[a, b]$ is a *step function* if there exist numbers $\{x_i\}_{i=0}^n$, with $x_0 = a < x_1 < \cdots < x_n = b$, such that f is constant on each open interval (x_{i-1}, x_i), namely there exist real numbers $\{y_i\}_{i=1}^n$, such that $f(x) = y_i$ for $x_{i-1} < x < x_i$.

Example 5.14. The function $f : [0, 1] \to \mathbf{R}$ defined by

$$f(x) = \begin{cases} 1/2^n & \text{if } 1/2^{n+1} < x \leq 1/2^n, \\ 0 & \text{if } x = 0, \end{cases}$$

is not technically a piecewise function or a step function because the domain has been divided into infinitely many subintervals.

Function Decompositions

Example 5.15. Let X be a non-empty set of real numbers, and let $f : X \to \mathbf{R}$ be a function. We define the *positive part* f^+ and *negative part* f^- of f by the formulas

$$\begin{aligned} f^+(x) &= \max\bigl(f(x), 0\bigr) = \frac{|f(x)| + f(x)}{2}, \\ f^-(x) &= -\min\bigl(f(x), 0\bigr) = \frac{|f(x)| - f(x)}{2}, \end{aligned}$$

compare the discussion of absolutely summable sequences in Chapter 4. Each function is non-negative.

Example 5.16. Let X be a set of real numbers that is "symmetric with respect to 0", i.e., $-X = X$. (See Definition 3.24.) A function $g : X \to \mathbf{R}$ is said to be *even* if $g(-x) = g(x)$ for all x in X. A function $g : X \to \mathbf{R}$ is said to be *odd* if $g(-x) = -g(x)$ for all x in X.

Assume $f : X \to \mathbf{R}$ is an arbitrary function, and define

$$\begin{aligned} f_{\text{even}}(x) &= \tfrac{1}{2}\bigl(f(x) + f(-x)\bigr), \\ f_{\text{odd}}(x) &= \tfrac{1}{2}\bigl(f(x) - f(-x)\bigr). \end{aligned}$$

These functions are easily checked to be even and odd (respectively), and to satisfy $f_{\text{even}}(x) + f_{\text{odd}}(x) = f(x)$. They are called the *even part* of f and the *odd part* of f.

Composition

Definition 5.17. Let X, Y, and Z be non-empty sets of real numbers, and let $f : X \to Y$ and $g : Y \to Z$ be functions. The *composition* of g and f is the function $g \circ f : X \to Z$ defined by

$$(g \circ f)(x) = g\big(f(x)\big) \quad \text{for all } x \text{ in } X.$$

Two functions g and f are *composable* if the target of f is contained in the domain of g.

Example 5.18. The polynomial functions $f(x) = x+1$ ("adding one") and $g(x) = x^2$ ("squaring") are composable (in either order), and

$$\begin{aligned}(g \circ f)(x) &= g(x+1) = (x+1)^2 = x^2 + 2x + 1,\\ (f \circ g)(x) &= f(x^2) = x^2 + 1.\end{aligned}$$

Note that the order of composition matters in general.

Example 5.19. Assume $f : X \to \mathbf{R}$. For each real number c, there is a function $\tau_c(z) = z - c$. The composition $g = f \circ \tau_c : c + X \to \mathbf{R}$ is defined by the formula $g(z) = f(z - c)$ for all z in $c + X$, i.e., by $g(c + x) = f(x)$ for all x in X.

Definition 5.20. Let $\ell > 0$ be a real number. A function $f : \mathbf{R} \to \mathbf{R}$ is said to be *periodic with period* ℓ, or ℓ*-periodic*, if $f(x + \ell) = f(x)$ for all real x.

Remark 5.21. By an easy inductively argument, if f is ℓ-periodic, then $f(x + n\ell) = f(x)$ for every integer n.

Example 5.22. The characteristic function $\chi_{\mathbf{Z}}$ is 1-periodic.

Example 5.23. Let $a < b$ be real, and put $\ell = b - a$. If $f : [a, b) \to \mathbf{R}$ is arbitrary, there exists a unique ℓ-periodic function that agrees with f on $[a, b)$. This periodic function is also denoted $f : \mathbf{R} \to \mathbf{R}$, and is said to be obtained by *extending by periodicity.*

5.2 The Action of Functions on Sets

Image and Preimage

Definition 5.24. Let $f : X \to Y$ be a function. If $A \subseteq X$, the set

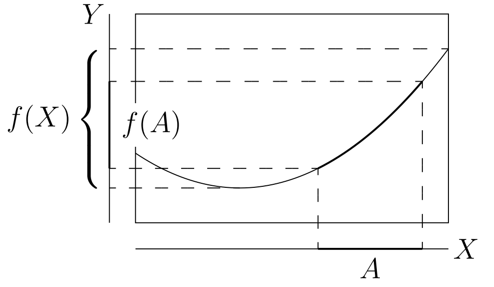
$$f(A) = \{y \text{ in } Y : y = f(x) \text{ for some } x \text{ in } A\}$$

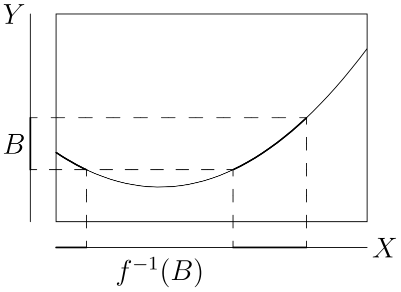
is called the *image of A under f*. In particular, the set $f(X)$ of values of f is called the *image* of f.

If $B \subseteq Y$, the set

$$f^{-1}(B) = \{x \text{ in } X : f(x) \in B\}$$

is called the *preimage of B under f*.





Inversion

Definition 5.25. Let $f : X \to Y$ and $g : Y \to X$ be functions. We say f and g are *inverse functions* if $(g \circ f)(x) = x$ for every x in X and $(f \circ g)(y) = y$ for every y in Y.

In this situation, we denote g by the symbol f^{-1}, read "f inverse". A function $f : X \to Y$ is said to be *invertible* if there exists an inverse $f^{-1} : Y \to X$.

Remark 5.26. A function has at most one inverse: If f is invertible, then $f^{-1}(y) = x$ if and only if $y = f(x)$, so the values of f uniquely determine the values of f^{-1}.

Procedurally, the inverse function "undoes" the action of f. For example, an equation $y = f(x)$ is "solved" for x by applying f^{-1} to both sides. The domain of f^{-1} is the target of f, and *vice versa.*

To say $f : X \to Y$ is invertible means that for every y in Y, there exists *exactly one* x in X such that $y = f(x)$. We then have a mapping $g : Y \to X$ defined by $g(y) = x$. Invertibility may therefore be separated into two useful criteria:

Definition 5.27. A function $f : X \to Y$ is *injective* if $f(x_1) = f(x_2)$ implies $x_1 = x_2$. Contrapositively, f is injective if $x_1 \neq x_2$ implies $f(x_1) \neq f(x_2)$, i.e., if f takes distinct values at distinct inputs.

A function $f : X \to Y$ is said to be *surjective* if for every y in Y, there exists an x in X such that $f(x) = y$. In words, every possible output value *is* actually a value of f.

Finally, f is *bijective* if f is both injective and surjective.

Remark 5.28. Let $f : X \to Y$ be a function, and assume $y \in Y$.

If f is injective, the equation $y = f(x)$ has *at most one* solution, i.e., the preimage $f^{-1}(\{y\})$ contains at most one element for each y.

If f is surjective, $y = f(x)$ has *at least one* solution, i.e., the preimage $f^{-1}(\{y\})$ is non-empty for each y.

Thus, f is bijective if and only if $y = f(x)$ has exactly one solution for every y in Y, if and only if f is invertible.

Example 5.29. Define $f : \mathbf{R} \to \mathbf{R}$ by $f(x) = x^2$. Suppose we wish to determine whether or not f is injective and/or surjective.

For injectivity, the general strategy is to assume $f(x_1) = f(x_2)$ and either deduce that $x_1 = x_2$, or find a specific pair of distinct inputs having equal output values.

Here, $x_1^2 = x_2^2$ if and only if

$$0 = x_1^2 - x_2^2 = (x_1 - x_2)(x_1 + x_2),$$

which implies $x_1 - x_2 = 0$ or $x_1 + x_2 = 0$. The second equation has solutions with $x_1 \neq x_2$, such as $x_1 = 1 = -x_2$, so f is not injective.

For surjectivity, let y be an arbitrary element of the target, and attempt to solve the equation $y = f(x)$ for x in the domain of f.

Here, we wish to solve $x^2 = y$ for x, with y an arbitrary real number. By Lemma 2.22 (iv), $x^2 \geq 0$ for all real x. Consequently, the equation $x^2 = -1$ has no real solution; that is, $y = -1$ is not in the image of f, so f is not surjective.

Example 5.30. Define $g : (0, \infty) \to \mathbf{R}$ by $g(x) = x^2$. Since the domain of g differs from the domain of f in the preceding example, g and f are *different functions* even though they are defined by the same formula.

Though g is not surjective for the same reasons as f above, g *is* injective: If $g(x_1) = g(x_2)$ for some *positive* real numbers x_1 and x_2, then either $x_1 - x_2 = 0$ or $x_1 + x_2 = 0$. But since x_1 and x_2 are positive, their sum is positive; thus $x_1 - x_2 = 0$, i.e., $x_1 = x_2$.

Example 5.31. Define $h : (0, \infty) \to (0, \infty)$ by $h(x) = x^2$. Since the target of h differs from the target of g in the preceding example, h and g

are *different functions* even though they have the same domain and are defined by the same formula.

The function h is injective for the same reasons as g. At this stage, however, it is unclear whether or not h is surjective. At issue is whether, for arbitrary real $y > 0$, the equation $x^2 = y$ has a positive, real solution x.

In school, you learned to write "$x = \sqrt{y}$", but this sidesteps the fundamental issue of whether "$\sqrt{y}$" is mathematically meaningful. (After all, "$\sqrt{-1}$" is *not* meaningful in the real number system.) It does turn out that h is surjective, see Theorem 6.53.

Definition 5.32. Let X be a non-empty set of real numbers. A function $f : X \to \mathbf{R}$ is said to be *strictly increasing* on X if the following holds:

If $x < x'$ are elements of X, then $f(x) < f(x')$.

Lemma 5.33. *Let $n \geq 1$ be an integer. The function $f : (0, \infty) \to \mathbf{R}$ defined by $f(x) = x^n$, is strictly increasing.*

Proof. Use Theorem 2.18 (iii) and induction on n. □

Remark 5.34. As with sequences, there is a small raft of analogous definitions for functions that are "non-decreasing", "strictly decreasing", and so forth. For example, f is *non-decreasing* if the following holds:

If $x < x'$ are elements of X, then $f(x) \leq f(x')$.

(Since the conclusion is automatic when $x = x'$, the hypothesis may be expressed freely as $x < x'$ or $x \leq x'$.)

Computationally, an increasing function may be applied to both sides of a *strict* inequality, preserving the sense and strictness of the inequality. A non-decreasing function may be applied to both sides of an inequality, preserving the sense of the inequality.

Remark 5.35. A strictly increasing function is injective: If $x_1 \neq x_2$, then without loss of generality we have $x_1 < x_2$, so $f(x_1) < f(x_2)$.

Similarly, a strictly decreasing function is injective.

Example 5.36. The reciprocal function $f(x) = 1/x$, defined for $x \neq 0$, is invertible (indeed, $f^{-1} = f$), hence injective. However, f is neither strictly increasing nor strictly decreasing: $-1 < 1$ and $1 < 2$, but $f(-1) < f(1)$ (so f is not decreasing) and $f(2) < f(1)$ (so f is not increasing).

Lemma 5.37. *Let $f : X \to Y$ be strictly increasing and surjective. The inverse function $f^{-1} : Y \to X$ is strictly increasing.*

Proof. Let $y < y'$ be arbitrary elements of Y, and let $x = f^{-1}(y)$ and $x' = f^{-1}(y')$. If x' were less than x, then y' would be less than y because f is increasing; consequently, we have $x < x'$, i.e., $f^{-1}(y) < f^{-1}(y')$. Since $y < y'$ were arbitrary, f^{-1} is increasing. □

Remark 5.38. An entirely similar argument shows that if f is strictly decreasing, then f^{-1} is strictly decreasing.

5.3 Cardinality

Definition 5.39. Two non-empty sets X and Y are said to have the *same cardinality* if there exists a bijection $f : X \to Y$.

Example 5.40. Two finite sets have the same cardinality if and only if they contain the same number of elements. The act of "counting" elements of a set X of n elements determines a bijection from X to the "initial sequence" $Y = \{1, 2, \dots, n\}$ in the set $\mathbf{Z}^+$ of positive integers.

Remark 5.41. A finite set X does not have the same cardinality as any proper subset Y. However, every infinite set *does* have the same cardinality as some proper subset, as examples below demonstrate.

Example 5.42. The set $\mathbf{N}$ of natural numbers has the same cardinality as the following proper subsets:

The set $n + \mathbf{N} = \{m \text{ in } \mathbf{N} : m \geq n\} = \{n, n+1, n+2, \dots\}$, with $n \geq 1$ an arbitrary natural number. The function $f : \mathbf{N} \to n + \mathbf{N}$ defined by $f(k) = n + k$ is a bijection, with inverse $g(m) = m - n$.

The set $2\mathbf{N} = \{0, 2, 4, \dots\}$ of even natural numbers. The function $f(k) = 2k$ is a bijection from $\mathbf{N}$ to $2\mathbf{N}$.

The set $2\mathbf{N}+1 = \{1, 3, 5, \dots\}$ of odd natural numbers. The function $f(k) = 2k + 1$ is a bijection from $\mathbf{N}$ to $2\mathbf{N} + 1$.

Example 5.43. The sets $\mathbf{N}$ and $\mathbf{Z}$ have the same cardinality. The function

$$f(n) = \begin{cases} n/2 & \text{if } n = 2k \text{ is even,} \\ -(1+n)/2 & \text{if } n = 2k+1 \text{ is odd,} \end{cases}$$

is a bijection from $\mathbf{N}$ to $\mathbf{Z}$. The set of even elements $2\mathbf{N}$ is mapped bijectively by f to the non-negative integers, and the complementary set $2\mathbf{N} + 1$ is mapped bijectively by f to the negative integers.

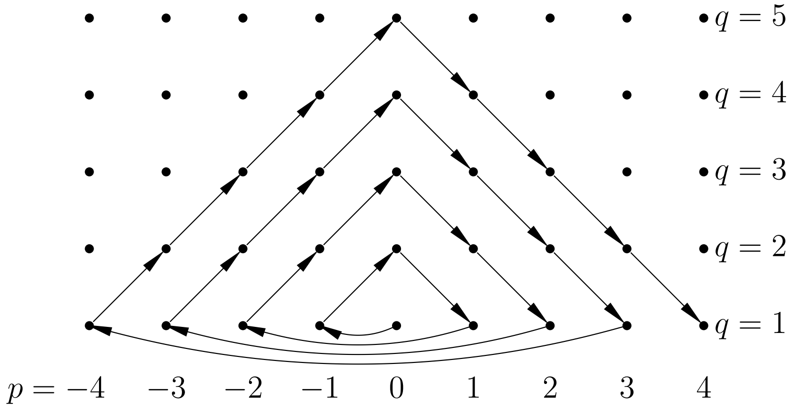
Example 5.44. Let $a < b$ be real numbers. The open intervals $(0, 1)$ and (a, b) are in bijective correspondence under

$$f(t) = a + t(b - a) = (1 - t)a + tb, \quad 0 < t < 1.$$

The inverse mapping is $g(u) = (u - a)/(b - a)$ for $a < u < b$. Consequently, any two bounded open intervals have the same cardinality. (The same formulas define bijections between the closed intervals $[0, 1]$ and $[a, b]$.)

The formula $f(t) = 1/t$ defines a bijection from $(0, 1)$ to $(1, \infty)$.

Example 5.45. The sets $\mathbf{N}$ and $\mathbf{Q}$ have the same cardinality. Each ordered pair (p, q) consisting of an integer p and a positive integer q is associated to a rational number, $r = p/q$. The set of all such pairs may be ordered in a list as indicated below.



To obtain a bijection from $\mathbf{N}$ to $\mathbf{Q}$, omit each pair (p, q) having a common factor greater than 1. The first ten values of this mapping are

$$(0, 1) \leftrightarrow 0, \quad (-1, 1) \leftrightarrow -1, \quad (1, 1) \leftrightarrow 1,$$
$$(-2, 1) \leftrightarrow -2, \quad (-1, 2) \leftrightarrow -\tfrac{1}{2}, \quad (1, 2) \leftrightarrow \tfrac{1}{2}, \quad (2, 1) \leftrightarrow 2,$$
$$(-3, 1) \leftrightarrow -3, \quad (-1, 3) \leftrightarrow -\tfrac{1}{3}, \quad (1, 3) \leftrightarrow \tfrac{1}{3}, \quad (3, 1) \leftrightarrow 3, \ \ldots.$$

Definition 5.46. A set X is *countable* or *countably infinite* if X has the same cardinality as $\mathbf{N}$, the set of natural numbers. We say X is *at most countable* if X is either finite or countable.

Remark 5.47. The preceding examples show that the set $\mathbf{Z}$ of integers and the set $\mathbf{Q}$ of rational numbers are countable. Modifications of the arguments given show that a countable union of countable sets is countable. The set of ordered pairs (p, q) in Example 5.45 may be viewed as the union over $q \geq 1$ of disjoint "copies" of the integers.

Example 5.48. The set of polynomials with rational coefficients is countable. A linear polynomial $a_0 + a_1 x$ with rational coefficients corresponds to a unique ordered pair (a_0, a_1) of rational numbers, so there are countably many such polynomials by the preceding remark.

Inductively, a polynomial of degree $n+1$ corresponds to a rational number (the top-degree coefficient) and a polynomial of degree n (the remaining terms). The set of polynomials of degree $n+1$ with rational coefficients is therefore a countable union of countable sets, and hence is itself countable.

Since there are only countably many possible degrees for a polynomial, the set of *all* polynomials (of arbitrary degree) with rational coefficients is a countable set.

Definition 5.49. A real number x is said to be *algebraic* if there exists a polynomial f with integer coefficients such that $f(x) = 0$. A non-algebraic real number is said to be *transcendental*.

Example 5.50. Every rational number is algebraic: If $r = p/q$ is rational, the linear polynomial $f(x) = qx - p$ has r as a root.

Similarly, square roots of positive rational numbers are algebraic; if $f(x) = qx^2 - p$, then $r = \sqrt{p/q}$ is a root (granted that the square root represents a real number).

Since a polynomial of degree n has at most n distinct real roots and the set of polynomials with integer coefficients is countable, there are only countably many algebraic numbers.

Not all infinite sets are countable. Particularly, the set $\mathbf{R}$ of real numbers is uncountable. We prove this in two stages:

Example 5.51. The set $X = \{(a_k) : a_k = 0 \text{ or } 1\}$ of all (countably-infinite) integer sequences whose terms are all 0 or 1 is uncountable. Our method of proof is the *Cantor diagonal argument.* We will show that if f is an arbitrary mapping from $\mathbf{N}$ to X, then f is not surjective; there exists a sequence not in the image of f.

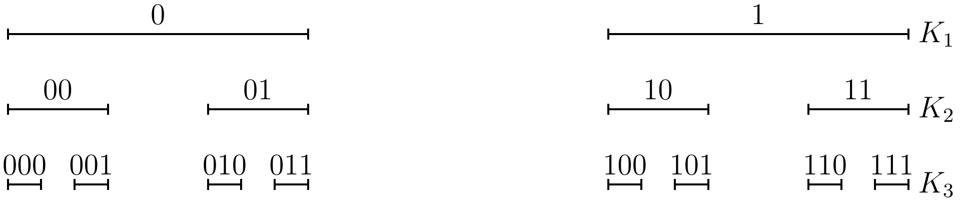
If $f : \mathbf{N} \to X$ is arbitrary, we may write $f(n) = (a_{nk})_{k=0}^{\infty}$ for each natural number n. Define a sequence $\mathbf{b}$ in X by the formula $b_k = 1 - a_{kk}$; that is, take the kth term of $\mathbf{b}$ to be the "opposite" of the kth term of the sequence $f(k)$. By construction, $\mathbf{b} \neq f(n)$ for every n; these sequences do not have the same nth term! In other words, $\mathbf{b}$ is not in the image of f. But $f : \mathbf{N} \to X$ was an arbitrary mapping; we conclude that no mapping from $\mathbf{N}$ to X is surjective.

Example 5.52. The set $\mathbf{R}$ is uncountable. In fact, the subset X' of $\mathbf{R}$ consisting of all infinite decimals whose digits are all either 0 or 1 is uncountable, because there is an obvious bijection between X' and the set X of binary sequences in the preceding example.

In Chapter 3 we saw that the set $\mathbf{Q}$ of rationals is dense in $\mathbf{R}$: Between any two real numbers, there is a rational number. We can now see a bit more deeply the complexity with which the rationals sit inside the reals, since the set of irrational numbers has larger cardinality than the set of rationals.

Example 5.53. The set of binary sequences is in bijective correspondence with the Cantor ternary set K, see Exercise 4.17. The geometric idea is simple. Recall that the Cantor set is the intersection of sets K_n, each obtained by removing the middle third of each subinterval of K_{n-1}. Assume $x \in K$, and imagine "zeroing in" on x by answering an infinite number of "yes-no" questions. The answer to each question tells us whether to pass to a left-hand or right-hand subinterval when we remove the middle third.

In more detail, we know initially that $x \in K_0 = [0, 1]$. If x is in $[0, \frac{1}{3}]$, set $a_0 = 0$; if x is in $[\frac{2}{3}, 1]$, set $a_0 = 1$. Inductively, assume we have constructed a finite sequence $a_0, \dots, a_{n-1}$, and we have located x in a particular component of K_n. This component is split into two subintervals in K_{n+1}. If x lies in the left subinterval, take $a_n = 0$; otherwise take $a_n = 1$.



Each x in K gives rise to a unique binary sequence, each binary sequence determines a unique element of K, and these associations are inverse to each other. It follows that K is uncountable, cf. Exercise 6.17.

Only countably many elements of K are endpoints of some K_n. Each set K_n has only finitely many endpoints. The set of all endpoints in K is therefore a countable union of finite sets.

Alternatively, the endpoints of K_n correspond to sequences that are constant after the nth term, see also Exercise 5.27, and there are only countably many such sequences. In any event, most elements of K are not endpoints of one of the sets K_n.

5.4 Power Series

"Power series" constitute one of the most important tools in elementary analysis. Loosely, a power series is "a polynomial with infinitely many summands".

Definition 5.54. Let $(a_k)_{k=0}^{\infty}$ be a real sequence, and let x_0 be a real number. The infinite series

$$\sum_{k=0}^{\infty} a_k(x - x_0)^k = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + a_3(x - x_0)^3 + \dots$$

is called the *power series* with *center* x_0 and *coefficients* (a_k).

Remark 5.55. Let I be the set of real x at which the preceding power series converges. When $x = x_0$ every term (except possibly the first) vanishes, and the series converges; that is, $x_0 \in I$. We will show presently (Theorem 5.63 below) that if the power series converges at some real number $x \neq x_0$, then the series converges *absolutely* at each point of the open ball with center x_0 and radius $|x - x_0|$; that is, I comprises either a single point or an interval centered at x_0 (possibly closed or half-open), called the *interval of convergence* of the power series. In I, the sum of the series defines a real-valued function f.

Example 5.56. The geometric series

$$\sum_{k=0}^{\infty} x^k = 1 + x + x^2 + x^3 + \dots$$

converges if and only if $|x| < 1$, see Example 4.67. That is, the interval of convergence is $I = (-1, 1)$, and

$$f(x) = \sum_{k=0}^{\infty} x^k = \frac{1}{1 - x}.$$

Replacing x by $-x$ gives the power series representation

$$\frac{1}{1 + x} = \frac{1}{1 - (-x)} = \sum_{k=0}^{\infty} (-x)^k = \sum_{k=0}^{\infty} (-1)^k x^k$$

on $(-1, 1)$.

Example 5.57. The algebraic identity

$$\frac{1}{1-x^2} = \frac{1}{2}\left(\frac{1}{1-x} + \frac{1}{1+x}\right), \quad x \neq \pm 1,$$

may be checked by putting the right-hand side over a common denominator. If $-1 < x < 1$, however, this identity has a remarkable interpretation in terms of geometric series:

$$\begin{aligned}
\frac{1}{1-x} &= 1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + \dots, \\
\frac{1}{1+x} &= 1 - x + x^2 - x^3 + x^4 - x^5 + x^6 - x^7 + \dots, \\
\frac{1}{1-x^2} &= 1 \qquad + x^2 \qquad + x^4 \qquad + x^6 \qquad + \dots;
\end{aligned}$$

the third line is half the term-by-term sum of the first two lines!

Generally, convergent power series can be added, subtracted, and multiplied by constants “just as if they were polynomials”, because of Theorem 4.64.

Example 5.58. Let $c \neq 0$. By Example 5.56,

$$\frac{1}{c-x} = \frac{1}{c\big(1-(x/c)\big)} = \frac{1}{c}\sum_{k=0}^{\infty}\left(\frac{x}{c}\right)^k = \sum_{k=0}^{\infty}\frac{x^k}{c^{k+1}}$$

provided $|x/c| < 1$, i.e., on the interval $I = (-|c|, |c|)$. Similarly,

$$\frac{1}{c+x} = \frac{1}{c}\sum_{k=0}^{\infty}(-1)^k\left(\frac{x}{c}\right)^k = \sum_{k=0}^{\infty}\frac{(-1)^k}{c^{k+1}}\,x^k \quad \text{on } I = (-|c|, |c|).$$

Definition 5.59. Let x_0 be a real number and $0 < R \leq \infty$. A real-valued function f defined on $B_R(x_0)$ is said to be *real-analytic* if f is represented on $B_R(x_0)$ by a convergent power series, i.e., there exist coefficients $(a_k)_{k=0}^{\infty}$ such that

$$f(x) = \sum_{k=0}^{\infty} a_k(x-x_0)^k \quad \text{for } |x-x_0| < R.$$

Remark 5.60. Our use of the term “real-analytic” is slightly stronger than the “standard” definition, but is convenient for our purposes.

Example 5.61. Let $x_0 \neq 0$ be arbitrary. The reciprocal function $f(x) = 1/x$ is real-analytic on $B_{|x_0|}(x_0)$. Indeed (compare Example 5.58 with $c = x_0$),

$$\frac{1}{x} = \frac{1}{x_0 + (x - x_0)} = \frac{1}{x_0} \sum_{k=0}^{\infty} (-1)^k \left(\frac{x - x_0}{x_0}\right)^k$$

provided $|(x - x_0)/x_0| < 1$, namely in the open ball of radius $|x_0|$ centered at x_0.

Remark 5.62. Suppose

$$g(x) = \sum_{k=0}^{\infty} a_k (x - x_0)^k = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \dots$$

converges on some open ball $B_R(x_0)$. Introducing $z = x - x_0$, we obtain a real-analytic function $f : (-R, R) \to \mathbf{R}$ defined by

$$f(z) = \sum_{k=0}^{\infty} a_k z^k = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \dots.$$

The domain of f is a symmetric interval, and $g(x) = f(x - x_0)$ for all x with $|x - x_0| < R$. Moreover, since z is a "dummy variable", we may as well call it x, and write $f(x) = \sum_k a_k x^k$.

The practical consequence is that in proving theorems about general power series, we may assume the center of the series to be 0, gaining notational simplicity without loss of generality.

Theorem 5.63. *Let $(a_k)_{k=0}^{\infty}$ be a real sequence, x_0 a real number. If the series $\sum_k a_k R^k$ converges for some $R \neq 0$, then the power series*

$$\sum_{k=0}^{\infty} a_k (x - x_0)^k$$

converges absolutely *for every x with $|x - x_0| < |R|$. That is, the interval of convergence I contains the open ball $B_{|R|}(x_0) = x_0 + (-|R|, |R|)$.*

Proof. By hypothesis, the sequence $(a_k R^k)$ is summable, hence convergent to 0 by the vanishing criterion (Theorem 4.66). Proposition 4.23 implies $(a_k R^k)$ is bounded; there exists a real number M such that

$$|a_k R^k| = |a_k| \cdot |R|^k \leq M \quad \text{for all } k \geq 0.$$

By Remark 5.62, we may assume $x_0 = 0$. Let x be an arbitrary number with $|x| < |R|$, and put $\rho = |x|/|R|$, so that $0 \leq \rho < 1$ and $|x| = |R| \cdot \rho$. We have

$$|a_k x^k| = |a_k| \cdot |x|^k = |a_k| \cdot |R|^k \cdot \rho^k \leq M\rho^k.$$

Since $M\rho^k$ is the general term of a convergent geometric series, $\big(|a_k x^k|\big)$ is summable by the comparison test. □

Corollary 5.64. *Let (a_k) be a real sequence, x_0 real. There exists an extended real number R with $0 \leq R \leq \infty$ such that the power series $\sum_k a_k(x - x_0)^k$ converges absolutely for all real x with $|x - x_0| < R$, and diverges for all x with $R < |x - x_0|$.*

Proof. Consider the set $J = \{x \text{ in } \mathbf{R} : \sum_k |a_k|\,|x|^k \text{ converges}\}$, and put $R = \sup J$. Since $0 \in J$, we have $0 \leq R$. Since $|-x| = |x|$ for all real x, the set J is symmetric about the origin.

Let x be a real number with $|x| < R = \sup J$. By the definition of a supremum, there exists an r in J such that $|x| < r$. By Theorem 5.63, $(-r, r) \subseteq J$. Since x in $(-R, R)$ is arbitrary, $(-R, R) \subseteq J$.

Conversely, if $R < |x| = r$, the power series $\sum_k a_k x^k$ diverges; if the series converged, Theorem 5.63 would imply that $(-r, r) \subseteq J$, contrary to the fact that $R = \sup J$. □

Definition 5.65. The extended real number R in Corollary 5.64 is called the *radius of convergence* of the power series $\sum_k a_k(x - x_0)^k$.

Remark 5.66. If $R = 0$, the condition $|x - x_0| < R$ is empty, and the power series diverges for all $x \neq x_0$. Similarly, if $R = \infty$, the condition $R < |x - x_0|$ is empty, and the series converges absolutely for all real x. The corollary gives no information about convergence when $|x - x_0| = R$.

Theorem 5.67. *Let $(a_k)_{k=0}^{\infty}$ be a sequence of coefficients. If*

$$|x - x_0| \lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| < 1,$$

i.e., the limit exists and is smaller than 1, then the power series

$$\sum_{k=0}^{\infty} a_k(x - x_0)^k$$

converges absolutely at x.

Proof. For each real x, a power series is an ordinary numerical series, with terms $b_k = a_k(x - x_0)^k$. By the ratio test, if $|b_{k+1}/b_k| \to \rho < 1$, the series converges absolutely. But by algebra,

$$\lim_{k\to\infty} \left|\frac{b_{k+1}}{b_k}\right| = \lim_{k\to\infty} \left|\frac{a_{k+1}(x - x_0)^{k+1}}{a_k(x - x_0)^k}\right| = |x - x_0| \lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right|. \qquad \square$$

Remark 5.68. The statement of Theorem 5.67 is convenient for many purposes, but the method of proof applies more generally to power series having some coefficients equal to 0. In general, let b_k denote the kth term (which is a function of x), and apply the ratio test to the series $\sum_k b_k$.

Example 5.69. The power series

$$\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

converges for all real x, i.e., $R = \infty$. Here, $b_k = (-1)^k x^{2k+1}/(2k+1)!$,

$$\begin{aligned} \lim_{k\to\infty} \left|\frac{b_{k+1}}{b_k}\right| &= \lim_{k\to\infty} \left|\frac{(-1)^{k+1}x^{2k+3}}{(2k+3)!} \cdot \frac{(2k+1)!}{(-1)^k x^{2k+1}}\right| \\ &= \lim_{k\to\infty} \left|\frac{x^2(2k+1)!}{(2k+3)!}\right| = \lim_{k\to\infty} \left|\frac{x^2}{(2k+3)(2k+2)}\right| = 0, \end{aligned}$$

and this is less than 1 for all real x.

Example 5.70. The power series

$$\sum_{k=1}^{\infty} x^{k!} = x + x^2 + x^6 + x^{24} + \dots$$

converges if and only if $|x| < 1$. First, if $|x| \ge 1$, the terms of the series do not approach 0, so the series certainly diverges. Otherwise, $b_k = x^{k!}$, so $|b_{k+1}/b_k| = |x^{(k+1)!-k!}|$. As $k \to \infty$, the exponent grows without bound, so $|x^{(k+1)!-k!}| \to 0$ by Theorem 4.18, and the power series converges by the ratio test.

Exercises

Exercise 5.1. Use the definition to find the even and odd parts of:
(a) $f(x) = 3 - 2x + 5x^2 + x^5$. (b) $g(x) = \chi_{[0,1]}(x)$.

Exercise 5.2. For each function f given, sketch the graphs $y = f(x)$, $y = f^+(x)$, and $f^-(x)$ on the same set of axes.
(a) $f(x) = x$. (b) $f(x) = |x| - 1$. (c) $f(x) = x(x-1)(x-2)$.

Exercise 5.3. Find the even and odd parts of $p(x) = x(x-1)^2$. Find the positive and negative parts of p; express your answer as a piecewise-polynomial function.

Exercise 5.4. Let $a < b$ be real numbers, and let $X = (a, b)\backslash\{0\}$ be the set of non-zero real numbers between a and b. Find, with justification, the image of X under the reciprocal function $f(x) = 1/x$, assuming:
(a) $b < 0$. (b) $b = 0$. (c) $a < 0 < b$. (d) $a = 0$. (e) $0 < a$.

Exercise 5.5. Construct two bijections between the intervals $(0, 1)$ and $(0, \infty)$, one increasing, the other decreasing. Suggestions: The rational function $f(t) = 1/t$ maps $(0, 1)$ bijectively to $(1, \infty)$, and $g(t) = 1 - t$ maps $(0, 1)$ bijectively to itself.

Exercise 5.6. Construct a bijection between the intervals $(-1, 1)$ and $(-\infty, \infty)$. (If necessary, you may use square roots to prove your function is surjective.)

Exercise 5.7. Let (a_k) be a real sequence. In each part, there exists a unique function $f : [0, \infty) \to \mathbf{R}$ satisfying the given condition as well as $f(k) = a_k$ for all natural numbers k. Write down "piecewise-type" formulas for f, and carefully sketch the graph of f on the interval $[0, 10]$ if $a_k = (-1)^k/(k + 1)$.

(a) The graph of f is a straight line segment on each interval $[k, k+1]$.

(b) f is constant on each interval $[k, k + 1)$.

(c) f is constant on each interval $(k, k + 1]$.

Exercise 5.8. Consider the rational function $f(x) = (1 + x)/(1 - x)$. Calculate the compositions $f \circ f$, $f \circ f \circ f$, and $f \circ f \circ f \circ f$, giving the domain of each, and simplifying formulas as much as possible.

Exercise 5.9. Let $f : X \to Y$ and $g : Y \to Z$ be bijections. Prove that the composition $g \circ f : X \to Z$ is a bijection, and express the inverse of $g \circ f$ in terms of the inverses of f and g.

Exercise 5.10. Let f be defined on all of $\mathbf{R}$.

(a) What simple geometric operation on the Cartesian plane converts the graph $y = \frac{1}{2}f(2x)$ into the graph $y = f(x)$?

(b) Generally, if $c > 1$, describe the graph $y = \frac{1}{c}f(cx)$.

Exercise 5.11. Let A and B be subsets of $\mathbf{R}$, and let χ_A and χ_B be their characteristic functions. Establish the following:

(a) $1 - \chi_A = \chi_{A^c}$, the characteristic function of A^c.

(b) $\min(\chi_A, \chi_B) = \chi_{A\cap B} = \chi_A \cdot \chi_B$.

(c) $\max(\chi_A, \chi_B) = \chi_{A\cup B} = \chi_A + \chi_B - \min(\chi_A, \chi_B)$.

Exercise 5.12. This exercise characterizes step functions.

(a) For $k = 1, \dots, n$, let I_k be an interval, χ_k the characteristic function of I_k, and c_k a real number. Use Exercise 5.11 and induction on n to prove that

$$f(x) = \sum_{k=1}^{n} c_k \chi_k(x) \tag{‡}$$

is a step function on $\mathbf{R}$. Sketch functions of the form $c_1\chi_1 + c_2\chi_2$ for which the sets I_1 and I_2 are, or are not, disjoint.

(b) Can every step function be written in the form (‡)? If so, prove it; if not, what properties of the sets I_k need to be modified?

(c) Is the representation from part (b) unique? If so prove it; if not, what properties of the sets I_k need to be modified?

Exercise 5.13. For each of the following power series, write out the first four non-zero terms and find the radius. If the radius is positive and finite, determine whether or not the series converges at either endpoint.

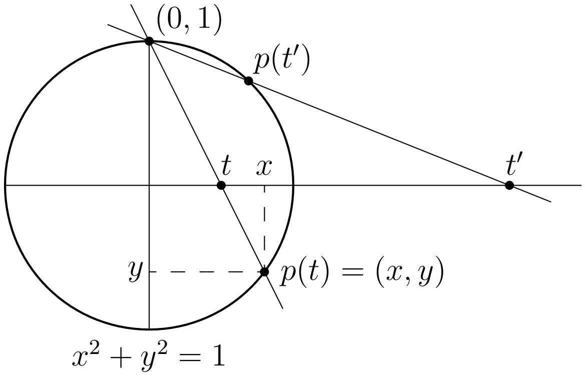
(a) $\displaystyle\sum_{k=1}^{\infty} \frac{x^k}{k^2}$. (b) $\displaystyle\sum_{k=1}^{\infty} \frac{x^{2k}}{k^2}$. (c) $\displaystyle\sum_{k=1}^{\infty} k^2 x^k$.

Exercise 5.14. Repeat the preceding exercise for:

(a) $\displaystyle\sum_{k=1}^{\infty} \frac{x^k}{k}$. (b) $\displaystyle\sum_{k=1}^{\infty} \frac{x^{2k}}{k}$. (c) $\displaystyle\sum_{k=1}^{\infty} k x^k$.

Exercise 5.15. Repeat the preceding exercise for:

(a) $\displaystyle\sum_{k=0}^{\infty} \frac{x^k}{k!}$. (b) $\displaystyle\sum_{k=0}^{\infty} k! x^k$. (c) $\displaystyle\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}$.



Exercise 5.16. Let X be the set of points $(x, y) \neq (0, 1)$ in the plane satisfying $x^2 + y^2 = 1$. Define a mapping $p : \mathbf{R} \to X$ by joining $(0, 1)$ to $(t, 0)$ by a straight line and letting $p(t) = (x, y)$ be the point of intersection with the circle, see diagram.

(a) Use similar triangles to find (x, y) in terms of t. Your answer should be a pair of rational functions of t.

(b) Show that p is bijective, both geometrically and by finding a formula for p^{-1}, i.e., expressing t in terms of x and y.

(c) Prove t is rational if and only if x and y are rational. Conclude that if A, B, and C are positive integers satisfying $A^2 + B^2 = C^2$, then there exist positive integers p and q such that $C = p^2 + q^2$, and $\{A, B\} = \{2pq, p^2 - q^2\}$, i.e., these two sets are the same.

(d) Let $f(t) = 1/t$ for $t \neq 0$. Show that $p^{-1} \circ f \circ p(x, y) = (x, -y)$, reflection of the circle across the x-axis.

(e) Let $f(t) = (1 + t)/(1 - t)$ for $t \neq -1, 0, 1$. Show similarly that $p^{-1} \circ f \circ p(x, y) = (-y, x)$, the counterclockwise quarter turn of the circle, compare Exercise 5.8.

Exercise 5.17. Let

$$\begin{aligned} p(x) &= a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0, \\ q(x) &= b_m x^m + b_{m-1} x^{m-1} + \cdots + b_1 x + b_0 \end{aligned}$$

be polynomials with real coefficients, and suppose $m \leq n$, i.e., the degree of q is less than or equal to the degree of p.

(a) Show that there exist polynomials $d(x)$ and $r(x)$ with real coefficients such that (i) $p(x) = d(x)q(x) + r(x)$ and (ii) the degree of $r(x)$

is strictly smaller than the degree of $q(x)$.

(b) Use part (a) to prove that if x_0 is real, then $p(x_0) = 0$ if and only if $(x - x_0)$ divides $p(x)$.

Exercise 5.18. Let p and q be polynomials of degree at most n and having real coefficients.

(a) Show that $p + q$ is a polynomial of degree at most n. If p and q both have degree n, does $p + q$ have degree n?

(b) Show that if c is a real number, then cp is a polynomial of degree at most n. Is the degree of cp always equal to n?

(c) Suppose $x_0 < x_1 < \cdots < x_n$ are real numbers, and $p(x_i) = 0$ for $i = 0, 1, \dots, n$. Prove that $p(x) = 0$ for all real x.
Hint: Use part (b) of the preceding exercise, and proceed by induction on n.

Let (x_1, y_1), (x_2, y_2), …, (x_n, y_n) be points in the plane, and assume $x_1 < x_2 < \cdots < x_n$. (That is, no two of the points lie on a vertical line; by renaming, we may assume the points are numbered from left to right.) The next few exercises outline the construction of the *Lagrange interpolation polynomial*, a unique polynomial of degree at most $n - 1$ whose graph contains all n points.

Exercise 5.19. Suppose $n = 2$; that is, we seek a polynomial of degree at most 1 whose graph (a line) passes through specified points (x_1, y_1) and (x_2, y_2) with $x_1 < x_2$.

(a) The formulas

$$e_1(x) = \frac{x - x_2}{x_1 - x_2}, \qquad e_2(x) = \frac{x - x_1}{x_2 - x_1}$$

define polynomials of degree 1. Show that $e_i(x_j) = 1$ if $i = j$ and $e_i(x_j) = 0$ if $i \neq j$.

That is, $e_1(x_1) = 1$ and $e_1(x_2) = 0$, while $e_2(x_1) = 0$ and $e_2(x_2) = 1$.

(b) Use Exercise 5.18 to prove the polynomial $p(x) = y_1 e_1(x) + y_2 e_2(x)$ has degree at most 1. Show that the points (x_1, y_1) and (x_2, y_2) satisfy $y = p(x)$.

Exercise 5.20. Suppose $n = 3$; that is, we seek a polynomial of degree at most 2 whose graph (a line or parabola) passes through specified points (x_1, y_1), (x_2, y_2), and (x_3, y_3) with $x_1 < x_2 < x_3$.

(a) The formulas

$$\begin{aligned} f_1(x) &= (x - x_2)(x - x_3), & e_1(x) &= \frac{f_1(x)}{f_1(x_1)}, \\ f_2(x) &= (x - x_1)(x - x_3), & e_2(x) &= \frac{f_2(x)}{f_2(x_2)}, \\ f_3(x) &= (x - x_1)(x - x_2), & e_3(x) &= \frac{f_3(x)}{f_3(x_3)}, \end{aligned}$$

define polynomials e_i of degree 2. Show that $e_i(x_j) = 1$ if $i = j$ and $e_i(x_j) = 0$ if $i \neq j$.

(b) Show the polynomial $p(x) = y_1 e_1(x) + y_2 e_2(x) + y_3 e_3(x)$ has degree at most 2, and that the points (x_1, y_1), (x_2, y_2), and (x_3, y_3) satisfy $y = p(x)$.

(c) Use parts (a) and (b) to find the unique quadratic polynomial $p(x)$ whose graph passes through the points $(-1, y_1)$, $(0, y_2)$, and $(1, y_3)$. Under what conditions does $p(x)$ have degree less than 2?

Exercise 5.21. Let (x_1, y_1), (x_2, y_2), …, (x_n, y_n) be points in the plane, and assume $x_1 < x_2 < \cdots < x_n$.

(a) For each $i = 1, \dots, n$, the formulas

$$f_i(x) = (x - x_1) \dots (x - x_{i-1})(x - x_{i+1}) \dots (x - x_n), \quad e_i(x) = \frac{f_i(x)}{f_i(x_i)},$$

define polynomials e_i of degree $(n-1)$. Show that $e_i(x_j) = 1$ if $i = j$ and $e_i(x_j) = 0$ if $i \neq j$.

(b) Show the polynomial $p(x) = y_1 e_1(x) + y_2 e_2(x) + \cdots + y_n e_n(x)$ has degree at most $(n - 1)$, and the points (x_i, y_i) satisfy $y = p(x)$.

(c) Suppose $y = q(x)$ is a polynomial of degree at most $(n - 1)$ and $y_i = q(x_i)$ for $i = 1, \dots, n$. Prove that $q(x) = p(x)$ for all real x. Hint: Apply Exercise 5.18 (c) to the polynomial $p(x) - q(x)$.

Exercise 5.22. Let $f : \mathbf{R} \to \mathbf{R}$ be ℓ-periodic for some $\ell > 0$. Prove that the even and odd parts of f are also ℓ-periodic.

Exercise 5.23. Let $\ell > 0$. A function $f : \mathbf{R} \to \mathbf{R}$ is *ℓ-anti-periodic* if $f(x+\ell) = -f(x)$ for all real x. Prove that such a function is 2ℓ-periodic.

Exercise 5.24. Let X be a non-empty set of real numbers satisfying $-X = X$, and let f be a real-valued function on X.

Suppose there exists an even function E and an odd function O on X satisfying $f(x) = E(x) + O(x)$ for all x in X. Find formulas for $E(x)$ and $O(x)$ in terms of f. Use Example 5.16 to conclude that every function on X may be written *uniquely* as the sum of an even function and an odd function.

Exercise 5.25. Let $p(x)$ be a polynomial.

(a) Let $n \geq 0$ be an integer, and let $f(x) = x^n$. Show that f is an even function if and only if n is an even integer, and show $f(x)$ is odd if and only if n is an odd integer.

(b) Prove that p is an even function if and only if every summand has even degree, i.e., if and only if there exists a polynomial q such that $p(x) = q(x^2)$ for all real x.
Hint: You have formulas for decomposing $p(x)$ as the sum of an even and odd function, and by the preceding question this decomposition is unique.

(c) Prove that p is an odd function if and only if every summand has odd degree, i.e., if and only if there exists a polynomial q such that $p(x) = xq(x^2)$ for all real x.

(d) Generalize the conclusions of parts (b) and (c) to convergent power series whose domain is an interval symmetric about 0.

Exercise 5.26. Let X be a non-empty set of real numbers such that $-X = X$, and let f and g be functions on X.

(a) Prove that if f and g are both even or both odd, then fg is even.

(b) Prove that if f is even and g is odd, then fg is odd.

(c) Suppose f is even, and h is an arbitrary function defined on all of $\mathbf{R}$. Is either composition $f \circ h$ or $h \circ f$ necessarily even? Give a proof or counterexample.

Exercise 5.27. Let K be the Cantor set. By Exercise 4.17, there is a bijection between K and the set of sequences $(a_k)_{k=1}^{\infty}$ with $a_k = 0$ or 2 for all k, since each real number x in K is expressed uniquely in the form

$$x = \sum_{k=1}^{\infty} \frac{a_k}{3^k} = \frac{a_1}{3} + \frac{a_2}{3^2} + \frac{a_3}{3^3} + \dots, \qquad a_k = 0 \text{ or } 2.$$

Show that x is an endpoint of the set K_n if and only if the corresponding sequence (a_k) is eventually constant; specifically, $a_k = 0$ for all $k \geq n$ or $a_k = 2$ for all $k \geq n$.

Exercise 5.28. In the notation of the preceding exercise, let x in K correspond to the sequence (a_k) of 0s and 2s. Set $b_k = a_k/2$ (so that (b_k) is a sequence of 0s and 1s), consider the infinite series

$$y = \sum_{k=1}^{\infty} \frac{b_k}{2^k} = \frac{b_1}{2} + \frac{b_2}{2^2} + \frac{b_3}{2^3} + \dots,$$

and show that y is the general element of $[0, 1]$. That is, there is a surjective mapping $f : K \to [0, 1]$.

Exercise 5.29. Every real number x has a unique non-terminating decimal representation. (For example, $1/2 = 0.4\overline{9}$.) The decimal of x contains the digit 0 either finitely many times, or infinitely many times, depending on x. Define the function $f : \mathbf{R} \to \mathbf{R}$ by

$$f(x) = \begin{cases} n & \text{if the decimal of } x \text{ contains exactly } n \text{ zeros,} \\ -1 & \text{if the decimal of } x \text{ contains infinitely many zeros.} \end{cases}$$

Prove that f is unbounded, and achieves the value -1, on *every* open interval.

Exercise 5.30. Let $A = \mathbf{Q} \cap [0, 1]$ be the set of rational numbers between 0 and 1 (inclusive).

(a) Construct a sequence (a_n) such that every element of A is a term of the sequence.

(b) Prove that if $x \in [0, 1]$, then there exists a subsequence $(a_{\nu(k)})$ that converges to x.

Chapter 6

Continuous Functions

Throughout this chapter, X denotes a non-empty set of real numbers.

Definition 6.1. Let $f : X \to \mathbf{R}$ be a function, and $\mathbf{x} = (x_k)$ a sequence in X, i.e., a real sequence such that $x_k \in X$ for all k. We define the *image sequence* $\mathbf{y} = (y_k)$ by setting $y_k = f(x_k)$ for each k, namely, by applying f to the terms of (x_k).

Example 6.2. Suppose $f(x) = 1/x$ for $x \neq 0$.
If $x_k = (k+1)/k$ for $k \geq 1$, then $y_k = k/(k+1)$.
If $x_k = 1/k$, then $y_k = k$.
If $x_k = (-1)^k/k$, then $y_k = (-1)^k k$.

Example 6.3. The *signum* function $\operatorname{sgn} : \mathbf{R} \setminus \{0\} \to \mathbf{R}$ is defined by

$$\operatorname{sgn}(x) = \frac{x}{|x|} = \begin{cases} 1 & \text{if } 0 < x, \\ -1 & \text{if } x < 0. \end{cases}$$

If $x_k = 1/k$ for $k \geq 1$, then $y_k = 1$.
If $x_k = (-1)^k/k$, then $y_k = (-1)^k$.

6.1 Continuity

Definition 6.4. Let $f : X \to \mathbf{R}$ be a function, and assume $x \in X$. We say f is *continuous at* x if the following condition holds:

> For *every* sequence (x_k) in X that converges to x, the image sequence $\big(f(x_k)\big)$ converges to $f(x)$.

If f is continuous at x for each x in X, we say f is *continuous on* X.

Remark 6.5. Computationally, f is continuous at x if and only if

$$\lim_{k\to\infty} f(x_k) = f(x) = f\bigl(\lim_{k\to\infty} x_k\bigr)$$

whenever (x_k) is a sequence in X and $(x_k) \to x$.

Remark 6.6. If $f : X \to \mathbf{R}$ is continuous on X, and if A is a non-empty subset of X, then f is continuous on A, since every sequence in A is a sequence in X.

The following trivial observation is useful enough to state explicitly.

Lemma 6.7. *The identity function is continuous on* $\mathbf{R}$. *Every constant function is continuous on* $\mathbf{R}$.

Theorem 6.8. *Let f and g be continuous functions on X. The functions $f \pm g$ and fg are continuous on X. If Z is the set of x in X such that $g(x) = 0$, then the quotient f/g is continuous on $X \setminus Z$.*

Proof. This amounts to little more than a restatement of Theorem 4.31. In detail, let x be an arbitrary element of X, and let (x_k) be a sequence in X converging to x. We have, for example,

$$\begin{aligned}
\lim_{k\to\infty}(f+g)(x_k) &= \lim_{k\to\infty}\bigl[f(x_k)+g(x_k)\bigr] && \text{Definition of } f+g \\
&= \lim_{k\to\infty} f(x_k) + \lim_{k\to\infty} g(x_k) && \text{Theorem 4.31 (i)} \\
&= f(x) + g(x) && f,\ g \text{ continuous at } x \\
&= (f+g)(x).
\end{aligned}$$

The remaining assertions are proven entirely similarly. □

Corollary 6.9. *Let p and q be polynomials, $X = \{x \text{ in } \mathbf{R} : q(x) \neq 0\}$. The rational function $f = p/q : X \to \mathbf{R}$ is continuous on X. In particular, every polynomial function is continuous on* $\mathbf{R}$.

Proof. Intuitively, polynomials are obtained from the identity function and constant functions by adding and multiplying, so all polynomials are continuous.

Formally, we prove that polynomials are continuous by induction on the degree. Consider the statement $P(n)$:

> Every polynomial function of degree at most n is continuous on $\mathbf{R}$.

The base case $P(0)$ is true because polynomials of degree 0, i.e., constant functions, are (trivially) continuous. Assume inductively that for some $m \geq 0$ every polynomial of degree at most m is continuous. If $a_0, \dots, a_m, a_{m+1}$ are arbitrary real numbers, then the polynomial function

$$\begin{aligned} f(x) &= a_0 + a_1x + a_2x^2 + \cdots + a_mx^m + a_{m+1}x^{m+1} \\ &= a_0 + x(a_1 + a_2x + \cdots + a_mx^{m-1} + a_{m+1}x^m) \end{aligned}$$

is continuous: The term in parentheses is a polynomial of degree at most m, hence continuous by the inductive hypothesis, so f is a sum of products of continuous functions, and therefore continuous. We have shown that $P(m)$ implies $P(m+1)$ for all $m \geq 0$, so by the principle of mathematical induction, $P(n)$ is true for all $n \geq 0$.

By Theorem 6.8, a quotient of polynomials is continuous wherever the denominator is non-vanishing. □

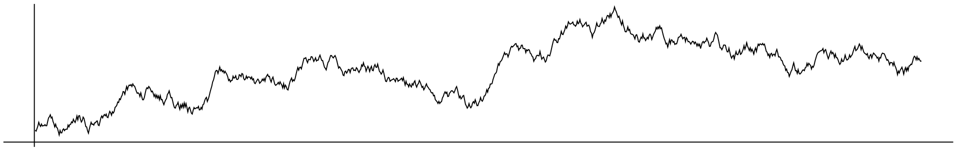
Proposition 6.10. *Let g and f be composable functions, and assume that f is continuous at x and g is continuous at $y = f(x)$. Then the composition $g \circ f$ is continuous at x.*

Proof. Let (x_k) be an arbitrary sequence in the domain of f that converges to x, and let $(y_k) = \big(f(x_k)\big)$ be the image sequence. Since f is continuous at x, the image sequence converges to $y = f(x)$. But since g is continuous at y, the sequence $\big(g(y_k)\big) = \big((g \circ f)(x_k)\big)$ converges to $g(y) = (g \circ f)(x)$. Since (x_k) was arbitrary, $g \circ f$ is continuous at x. □

Conceptualizing Continuous Functions

In calculus, continuous functions are normally depicted as "smooth" graphs with finitely many corners or cusps. "Zooming in" on most points causes the graph to "flatten out" into a line. As we will see in Chapter 8, this behavior corresponds to *differentiability.*

By contrast, a "typical" continuous function has a graph qualitatively similar to the accompanying diagram, more like the tracing of an



electrocardiogram, a seismograph, or stock price tracker.

Imagine the graph as a mountain range. Zooming in reveals cliffs and crags, then surfaces of rocks, which are "rough" at scales visible to the naked eye and do not appear more smooth when viewed under a microscope.

Mathematically, a "typical" continuous function possesses detail at arbitrarily small scales. Since the graph does not become more line-like upon zooming in on an arbitrary point, a typical continuous function is *nowhere differentiable*.

The full ramifications of these qualitative observations impose an intuition of continuous functions almost completely different from the picture presented in calculus. For instance, the graph of a typical continuous function is so rough that its length between *an arbitrary pair of points* is infinite. The only reason we can pretend to draw such a function is that a drawing instrument such as chalk, a laser printer, or a sharp pencil, does not delimit a mathematical point, but a small region. Our drawing instruments "cannot see" details smaller than the diameter of the tip, and they manage to "cover" the graph with a region of finite area, which can be "painted" in finite time.

The theorems in this chapter and the next apply to general continuous functions. For visual simplicity, proofs are illustrated using "smooth" functions. To sharpen your understanding, it's a good idea to ponder how the same arguments succeed against a general continuous function.

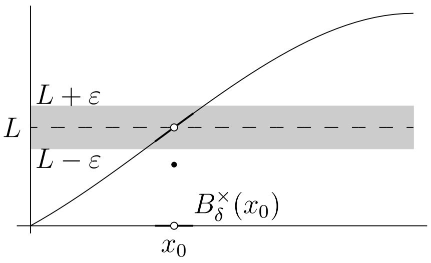
6.2 Limits

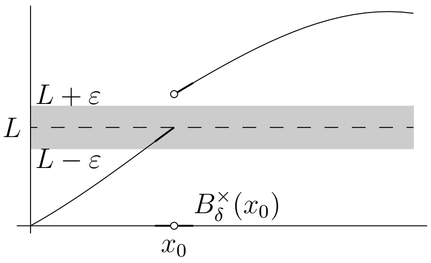
For theoretical work, it is convenient to have a "challenge-response" criterion for continuity that does not explicitly refer to arbitrary sequences. Recall that when $X \subseteq \mathbf{R}$, a *limit point* of X is a real number x_0 (possibly not an element of X) such that $B_\delta^\times(x_0) \cap X \neq \varnothing$ for every $\delta > 0$.

Definition 6.11. Let $f : X \to \mathbf{R}$ be a function, and x_0 a limit point of X. We say the real number L is a *limit* of f at x_0 if the following condition holds:

> For every $\varepsilon > 0$, there exists a $\delta > 0$ such that if x is an element of $B_\delta^\times(x_0) \cap X$, i.e., $x \in X$ and $0 < |x - x_0| < \delta$, then $|f(x) - L| < \varepsilon$.

Remark 6.12. Just as with sequential limits, the preceding definition may be regarded as an adversarial game between Player ε, who issues a challenge, and Player δ, who attempts to respond, making the distance from $f(x)$ to L smaller than ε merely by taking x within a distance δ of x_0. The limit of f at x_0 is L if and only if Player δ has a winning strategy against a perfect opponent (left-hand diagram).





As with a sequential limit, a functional limit is unique if it exists.

Lemma 6.13. *If L and L' are limits of f at x_0, then $L = L'$.*

Proof. Let $\varepsilon > 0$ be arbitrary. Because L and L' are limits of f at x_0, there exists a $\delta > 0$ such that if $x \in X$ and $0 < |x - x_0| < \delta$, then $|f(x) - L| < \varepsilon/2$ and $|f(x) - L'| < \varepsilon/2$. By the triangle inequality,

$$|L - L'| \leq |L - f(x)| + |f(x) - L'| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, $L = L'$. □

Remark 6.14. By the lemma, we are justified in treating a functional limit as a real number (provided the limit exists), and using notation such as

$$L = \lim(f, x_0) \quad \text{or} \quad L = \lim_{x \to x_0} f(x).$$

If $\lim(f, x_0)$ exists, we may express this fact by saying $f(x)$ *approaches L as x approaches x_0*, or by writing "$f(x) \to L$ as $x \to x_0$".

Though this "dynamical" language is convenient, it is arguably misleading. A functional limit at x_0 either exists or it doesn't; there is no contingency on x approaching the point x_0, and we do not have to "wait" (possibly forever) to find out whether or not the limit exists.

Theorem 6.15 (The sequential criterion for continuity)**.** *Let f be a real-valued function with domain X, and assume $x_\infty \in X$. The function f is continuous at x_∞ if and only if one of the following mutually exclusive conditions holds:*

(i) *x_∞ is an isolated point of X.*

(ii) x_∞ *is a limit point of* X*, and* $\lim(f, x_\infty) = f(x_\infty)$.

Proof. ((i) implies continuity). Suppose x_∞ is an isolated point of X. If (x_k) is a sequence in X that converges to x_∞, then (x_k) is eventually equal to x_∞, i.e., there is an index N such that $x_k = x_\infty$ for $k \geq N$. Continuity of f at x_0 follows immediately.

((ii) implies continuity). Suppose (ii) holds. Let $\varepsilon > 0$ be arbitrary. By (ii), there exists a $\delta > 0$ such that if $x \in X$ and $0 < |x - x_\infty| < \delta$, then $|f(x) - f(x_\infty)| < \varepsilon$. Since $|f(x_\infty) - f(x_\infty)| = 0 < \varepsilon$ automatically, f satisfies the slightly stronger predicate:

If $x \in X$ and $|x - x_\infty| < \delta$, then $|f(x) - f(x_\infty)| < \varepsilon$.

Let (x_k) be an arbitrary sequence in X that converges to x_∞; using the δ of the preceding paragraph as a “challenge”, there exists an index N such that if $k \geq N$, then $|x_k - x_\infty| < \delta$. But as just noted, this implies $|f(x_k) - f(x_\infty)| < \varepsilon$. Since $\varepsilon > 0$ was arbitrary, the image sequence $\big(f(x_k)\big)$ converges to $f(x_\infty)$; thus f is continuous at x_∞.

(Continuity implies (i) or (ii)). Contrapositively, we show that if both (i) and (ii) are false, then f is not continuous at x_∞.

Since (i) is false, x_∞ is a limit point of X. Since (ii) is false, Player ε has a winning strategy: There exists an $\varepsilon > 0$ such that no matter how $\delta > 0$ is chosen, there is some x in X with $|x - x_\infty| < \delta$ but $|f(x) - f(x_\infty)| \geq \varepsilon$.

Construct a sequence (x_k) as follows: For each positive integer k, pick x_k in X such that $|x_k - x_\infty| < 1/k = \delta$ but $|f(x_k) - f(x_\infty)| \geq \varepsilon$. The sequence (x_k) visibly converges to x_∞, but the image sequence $\big(f(x_k)\big)$ equally visibly does not converge to $f(x_\infty)$.

We have proven that if (i) and (ii) are false, then f is discontinuous at x_∞, which is the contrapositive of what was to be shown. □

Corollary 6.16 (The ε-δ criterion)**.** *Let* $f : X \to \mathbf{R}$ *be a function, and assume* $x_0 \in X$*. Then* f *is continuous at* x_0 *if and only if:*

For every $\varepsilon > 0$*, there exists a* $\delta > 0$ *such that if* $x \in X$ *and* $|x - x_0| < \delta$*, then* $|f(x) - f(x_0)| < \varepsilon$*.*

Corollary 6.17. *Let* $f : X \to \mathbf{R}$ *be continuous at some point* x_0 *in* X*.*

(i) *There exist positive real numbers* M *and* δ *such that* $|f(x)| < M$ *provided* $x \in X$ *and* $|x - x_0| < \delta$*.*

(ii) *If* $f(x_0) \neq 0$*, there is a* $\delta > 0$ *such that if* $x \in X$ *and* $|x - x_0| < \delta$*, then* $|f(x)| > |f(x_0)|/2$.

Remark 6.18. Conclusion (i) is expressed by saying that a continuous function is *locally bounded*. Conclusion (ii) says that if a continuous function is non-zero at some point, then the function is *locally bounded away from zero*; not merely is $|f(x)|$ non-zero, there is a *positive lower bound*, $|f(x_0)|/2$. In each case, a property of the function at one point "propagates" to some open interval.

Proof. (i) Take $\varepsilon = 1$. Let $M = |f(x_0)|+1$, and use continuity of f at x_0 to pick $\delta > 0$ such that if $x \in X$ and $|x-x_0| < \delta$, then $|f(x)-f(x_0)| < \varepsilon$. By the triangle inequality, if $x \in X$ and $|x - x_0| < \delta$, then

$$|f(x)| \leq |f(x_0)| + |f(x) - f(x_0)| < |f(x_0)| + 1 = M.$$

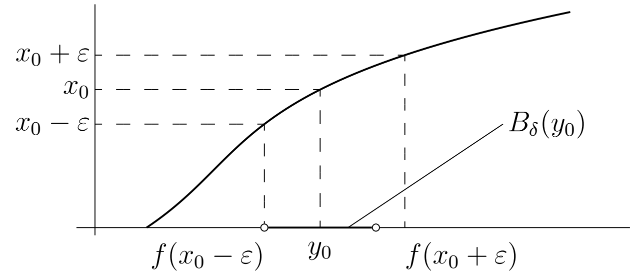
(i) Take $\varepsilon = |f(x_0)|/2$, which is positive by hypothesis, and use continuity of f at x_0 to pick $\delta > 0$ such that if $x \in X$ and $|x - x_0| < \delta$, then $|f(x) - f(x_0)| < \varepsilon = |f(x_0)|/2$. By the reverse triangle inequality, if $x \in X$ and $|x - x_0| < \delta$, then

$$|f(x)| \geq |f(x_0)| - |f(x) - f(x_0)| > |f(x_0)|/2. \qquad \square$$

Theorem 6.19. *Let* X *be an open interval of real numbers. If* f *is strictly monotone on* X*, then the inverse function* $f^{-1} : f(X) \to X$ *is continuous.*

Proof. Without loss of generality, we may assume f is strictly increasing. Fix y_0 in $f(X)$ and $\varepsilon > 0$ arbitrarily. There exists a unique x_0 in X such that $f(x_0) = y_0$. Since X is an open interval, we may assume (by making ε smaller if necessary) that $[x_0 - \varepsilon, x_0 + \varepsilon] \subseteq X$.

Since f is increasing, f^{-1} is increasing by Lemma 5.37, and we have $f(x_0 - \varepsilon) < y < f(x_0 + \varepsilon)$ if and only if $x_0 - \varepsilon < f^{-1}(y) < x_0 + \varepsilon$.



Put $\delta = \min\bigl(f(x_0+\varepsilon) - y_0, y_0 - f(x_0-\varepsilon)\bigr)$. We have $\delta > 0$, and if $y_0 - \delta < y < y_0 + \delta$, then $f(x_0-\varepsilon) < y < f(x_0+\varepsilon)$ by construction, so $x_0 - \varepsilon < f^{-1}(y) < x_0 + \varepsilon$.

That is, if $y \in f(X)$ and $|y - y_0| < \delta$, then $|f^{-1}(y) - f^{-1}(y_0)| < \varepsilon$. Since $\varepsilon > 0$ was arbitrary, f^{-1} is continuous at y_0. □

One-Sided Limits

Definition 6.20. Let X be a non-empty set of real numbers, and x_0 a real number. We say x_0 is *approachable from above* or *approachable from the right* in X if for every $\delta > 0$, there exists an x in X such that $x_0 < x < x_0 + \delta$; that is, $x_0 < x$ and $|x - x_0| < \delta$.

We say x_0 is *approachable from below* or *approachable from the left* in X if for every $\delta > 0$, there exists an x in X such that $x_0 - \delta < x < x_0$; that is, $x_0 > x$ and $|x - x_0| < \delta$.

Example 6.21. Let $X = (a, b)$. The left endpoint a is approachable from above in X, and the right endpoint b is approachable from below in X. The same is true if X is a closed or half-open interval.

Example 6.22. The point x_0 is approachable from both sides of the deleted neighborhood $B_r^\times(x_0)$.

Remark 6.23. A real number x_0 is a limit point of X if and only if x_0 is approachable from below or is approachable from above in X.

Definition 6.24. Let $f : X \to \mathbf{R}$ a function and x a real number.

If x is approachable from above in X, we say a real number L is a *limit from above* or a *limit from the right* of f at x if for every sequence (x_k) in X such that $x_k > x$ for all k, we have $\bigl(f(x_k)\bigr) \to L$.

If x is approachable from below in X, we say a real number L is a *limit from below* or a *limit from the left* of f at x if for every sequence (x_k) in X such that $x_k < x$ for all k, we have $\bigl(f(x_k)\bigr) \to L$.

Either type of limit is called a *one-sided limit* of f at x.

Proposition 6.25. *Let $f : X \to \mathbf{R}$ be a function and x_0 a limit point of X.*

(i) *If x_0 is approachable from above in X, and if L and L' are both limits of f at x_0 from above, then $L = L'$.*

(ii) *If x_0 is approachable from below in X, and if L and L' are both limits of f at x_0 from below, then $L = L'$.*

(iii) *If x_0 is approachable both from above and from below in X, then* $\lim(f, x_0)$ *exists and is equal to L if and only if both one-sided limits of f at x_0 exist and are equal to L.*

Remark 6.26. By (i) and (ii), one-sided limits are unique (if they exist). We write $L = \lim(f, x_0^+)$ to signify that L is the limit from above of f at x_0, and write $L = \lim(f, x_0^-)$ to indicate that L is the limit from below of f at x_0. These limits are alternatively denoted

$$\lim_{x \to x_0^+} f(x) = \lim(f, x_0^+), \qquad \lim_{x \to x_0^-} f(x) = \lim(f, x_0^-).$$

Proposition 6.27 (ε-δ criterion for one-sided limits)**.** *Let $f : X \to \mathbf{R}$ be a function, and assume x_0 is approachable from above in X. We have $L = \lim(f, x_0^+)$ if any only if the following holds:*

> *For every $\varepsilon > 0$, there exists a $\delta > 0$ such that if $x \in X$ and $x_0 < x < x_0 + \delta$, then $|f(x) - L| < \varepsilon$.*

Remark 6.28. There is an obvious modification for limits from below. These properties are established in the same way as the corresponding assertions for two-sided limits.

Theorem 6.29. *Let $f : X \to \mathbf{R}$ be bounded and monotone. For every real number x_0:*

(i) *If x_0 is approachable from above in X, then* $\lim(f, x_0^+)$ *exists.*

(ii) *If x_0 is approachable from below in X, then* $\lim(f, x_0^-)$ *exists.*

Remark 6.30. The boundedness hypothesis may be dropped, with the understanding that the one-sided limits of f at the "extremities" of X may be infinite.

Proof. (i) Assume x_0 is approachable from above in X. The set

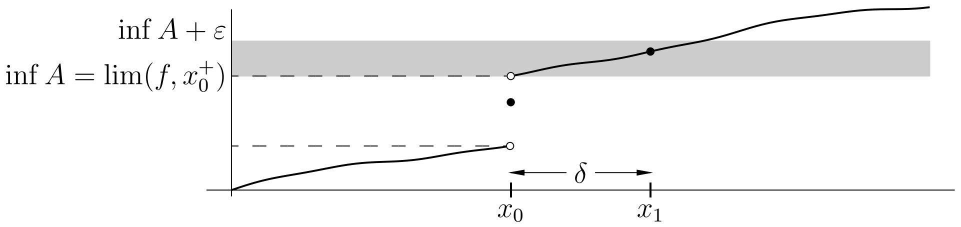
$$A = \{f(x) : x \in X \text{ and } x > x_0\} \neq \varnothing$$

is bounded below by $f(x_0)$. It suffices to show $\lim(f, x_0^+) = \inf A$.

Let $\varepsilon > 0$ be arbitrary. Since $\inf A + \varepsilon$ is not a lower bound of A, there exists an $x_1 > x_0$ in X such that $\inf A \leq f(x_1) < \inf A + \varepsilon$, see diagram. Put $\delta = x_1 - x_0 > 0$.

If $x_0 < x < x_0 + \delta = x_1$, then $\inf A \leq f(x) \leq f(x_1) < \inf A + \varepsilon$ by monotonicity. Rearranging, $0 \leq f(x) - \inf A < \varepsilon$, so $|f(x) - \inf A| < \varepsilon$. Since $\varepsilon > 0$ was arbitrary, $\lim(f, x_0^+) = \inf A$.

The proof of (ii) is entirely similar, see Exercise 6.3. □



The Calculi of Sloppiness

Analysis is founded on approximation and estimates. Computational notation for approximations will streamline our work.

Definition 6.31. Let f and g be functions on X, and assume $x_0 \in X$. We say g *approximates* f *to within* ε *near* x_0, and write

$$g(x) \approx f(x) + \mathsf{A}(\varepsilon) \quad \text{for } x \approx x_0,$$

if there exists a $\delta > 0$ such that $|g(x) - f(x)| < \varepsilon$ whenever $x \in X$ and $|x - x_0| < \delta$.

If g approximates f to within ε near x_0 for arbitrary $\varepsilon > 0$, we write $g(x) \approx f(x) + o(1)$ for $x \approx x_0$.

Example 6.32. If $f(x_0)$ is defined, then Corollary 6.16 asserts that $f(x) \approx f(x_0) + o(1)$ for $x \approx x_0$ if and only if f is continuous at x_0.

Example 6.33. Theorem 6.8 and the preceding example guarantee that "approximations" $f(x) \approx f(x_0) + o(1)$ and $g(x) \approx g(x_0) + o(1)$ for $x \approx x_0$ can be added, multiplied, and (if the denominator is non-zero) divided in the "obvious" way:

$$\begin{aligned} f(x) \pm g(x) &\approx f(x_0) \pm g(x_0) + o(1), \\ f(x) \cdot g(x) &\approx f(x_0) \cdot g(x_0) + o(1), \\ \frac{f(x)}{g(x)} &\approx \frac{f(x_0)}{g(x_0)} + o(1). \end{aligned}$$

In particular, $o(1) \pm o(1) \approx o(1)$, $o(1) \cdot o(1) \approx o(1)$, and $C \cdot o(1) \approx o(1)$ for arbitrary real C.

The notational benefit is clear; the "error term" $o(1)$ acts like a sponge, absorbing (and therefore hiding) "ignorable" cross terms. This magic works because we may shrink δ "inside $o(1)$", retaining an open interval on which the inequality of interest continues to hold.

Definition 6.34. Let g be a function that is non-zero on some deleted neighborhood of x_0. We write

$$f(x) \approx o\big(g(x)\big) \quad \text{for } x \approx x_0,$$

read "f is little-o of g near x_0", if $f(x)/g(x) \approx o(1)$ for $x \approx x_0$.

Remark 6.35. Intuitively, $f \approx o(g)$ for $x \approx x_0$ means $|f(x)|$ is infinitesimally small compared to $|g(x)|$ when x is sufficiently close to x_0.

Example 6.36. If $k \geq 0$ is an integer, then

$$\frac{x^{k+1}}{x^k} = x \approx o(1) \quad \text{for } x \approx 0,$$

so $x^{k+1} \approx o(x^k)$ for $x \approx 0$. More generally, if $m > 0$, then $x^{k+m} \approx o(x^k)$ for $x \approx 0$.

Similarly, if x_0 is an arbitrary real number and if $m > 0$, then $(x - x_0)^{k+m} \approx o(x - x_0)^k$ for $x \approx x_0$.

Definition 6.37. Let f and g be functions on X. We write

$$f(x) \approx O(1) \quad \text{for } x \approx x_0,$$

read "$f(x)$ is big-O of 1" near x_0, if there exist positive real numbers C and δ such that $|f(x)| \leq C$ provided $x \in X$ and $|x - x_0| < \delta$. That is, $|f|$ is locally bounded near x_0.

We write

$$f(x) \approx O\big(g(x)\big) \quad \text{for } x \approx x_0,$$

read "f is big-O of g", if $f/g \approx O(1)$ for $x \approx x_0$.

Example 6.38. If $f(x_0)$ is defined and $f(x) \approx f(x_0) + O(x - x_0)$ for $x \approx x_0$, then f is continuous at x_0: Indeed, there exist positive real numbers C and δ such that $|f(x) - f(x_0)| < C|x - x_0|$ provided $|x - x_0| < \delta$.

Generally, the following conditions (for $x \approx x_0$) are "increasingly strong", in that each is implied by the next, but not by the previous: $f(x) \approx O(1)$, $f(x) \approx o(1)$, $f(x) \approx O(x - x_0)$, $f(x) \approx o(x - x_0)$, $f(x) \approx O(x - x_0)^2$, $f(x) \approx o(x - x_0)^2$, $\dots$.

Remark 6.39. The little-o and big-O notations are widespread. By contrast, A notation, first advocated in a letter by D. Knuth published in the March 1998 *Notices of the American Mathematical Society*, is not in wide usage.

6.3 Properties and Applications

Continuity of Power Series

Theorem 6.40. *Let* $\sum_k a_k z^k$ *be a power series with radius* $R > 0$*, and let* x_0 *be a real number. The real-analytic function*

$$f(x) = \sum_{k=0}^{\infty} a_k (x - x_0)^k$$

is continuous on the interval $B_R(x_0) = x_0 + (-R, R)$.

Remark 6.41. As usual, replacing $x - x_0$ by x reduces the theorem to the case $x_0 = 0$.

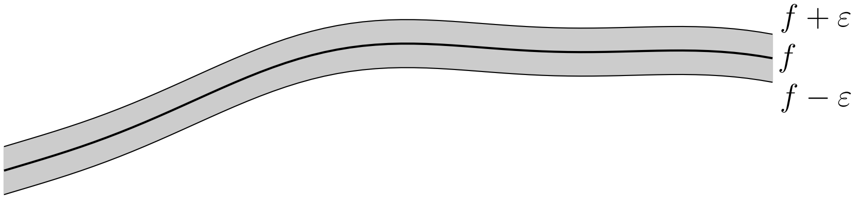
Conceptually, the proof proceeds by showing that (i) for each $r < R$, the sum f is approximated arbitrarily closely on $[-r, r]$ by its partial sums, and (ii) a function that is approximated arbitrarily closely by continuous functions is itself continuous.

Definition 6.42. Let f and g be functions on X. We say g *approximates* f *to within* ε *on* X, or g *uniformly approximates* f *to within* ε, and write

$$g \approx f + \mathsf{A}(\varepsilon) \quad \text{on } X,$$

if $|g(x) - f(x)| < \varepsilon$ for all x in X.

Remark 6.43. Geometrically, "$g \approx f + \mathsf{A}(\varepsilon)$ on X" means the graph of g over X lies in the "tube" of radius ε centered on the graph of f, i.e., $f(x) - \varepsilon < g(x) < f(x) + \varepsilon$ for all x in X.



Two functions f and g are equal on X if and only if $g \approx f + \mathsf{A}(\varepsilon)$ on X for every $\varepsilon > 0$.

Proposition 6.44. *Let* $f(x) = \sum_k a_k x^k$ *be a convergent power series with radius* $R > 0$*. For each* $n \geq 0$*, define*

$$f_n(x) = \sum_{k=0}^{n} a_k x^k = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n.$$

For every $\varepsilon > 0$ and every positive real number $r < R$, there exists an N such that if $n \geq N$, then $f \approx f_n + \mathsf{A}(\varepsilon)$ on $[-r, r]$.

Proof. Fix $\varepsilon > 0$ and $0 < r < R$ arbitrarily.

By Corollary 5.64, $\sum_k |a_k r^k|$ converges, so its tails converge to 0; specifically, there exists an N such that if $n \geq N$, then

$$\sum_{k=n+1}^{\infty} |a_k r^k| \leq \sum_{k=N+1}^{\infty} |a_k r^k| < \varepsilon.$$

But for all x in $[-r, r]$, we have

$$\begin{aligned} |f(x) - f_n(x)| = \left| \sum_{k=0}^{\infty} a_k x^k - \sum_{k=0}^{n} a_k x^k \right| &= \left| \sum_{k=n+1}^{\infty} a_k x^k \right| \\ \leq \sum_{k=n+1}^{\infty} |a_k x^k| &\leq \sum_{k=n+1}^{\infty} |a_k r^k| < \varepsilon; \end{aligned}$$

that is, $f \approx f_n + \mathsf{A}(\varepsilon)$ on $[-r, r]$. □

Proof of Theorem 6.40. Each partial sum f_n of the series for f is a polynomial, hence continuous on $(-R, R)$.

Fix x in $(-R, R)$ and $\varepsilon > 0$ arbitrarily. Set $r = \frac{1}{2}(|x| + R)$, so that $|x| < r < R$, and use Proposition 6.44 to choose N such that

$$f \approx f_N + \mathsf{A}(\varepsilon/3) \quad \text{on } [-r, r].$$

Now use continuity of f_N to choose a positive number $\delta < r - |x|$ such that if $|t - x| < \delta$, then $|f_N(t) - f_N(x)| < \varepsilon/3$.

If $|t - x| < \delta$, then $|t| \leq |x| + |t - x| < |x| + \delta < r$. That is, $B_\delta(x) \subseteq [-r, r]$. By the triangle inequality, if $|t - x| < \delta$, then

$$|f(t) - f(x)| \leq \underbrace{|f(t) - f_N(t)|}_{<\varepsilon/3} + \underbrace{|f_N(t) - f_N(x)|}_{<\varepsilon/3} + \underbrace{|f_N(x) - f(x)|}_{<\varepsilon/3} < \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, f is continuous at x. Since x in $(-R, R)$ was arbitrary, f is continuous on the open interval $(-R, R)$. □

Theorem 6.45 (The identity theorem for power series)**.** *Let*

$$f(x) = \sum_{k=0}^{\infty} a_k (x - x_0)^k$$

be a convergent power series with radius $R > 0$. If $f(x) = 0$ for all x in $B_R(x_0)$, then $a_k = 0$ for all k.

Remark 6.46. Theorem 6.45 guarantees that if two convergent power series centered at x_0 agree on some open interval I, they are identical (i.e., have the same coefficients).

Proof. As usual, assume $x_0 = 0$. We will establish the contrapositive: If some coefficient is non-zero, then $f(x)$ is not identically zero. Let n be the smallest index such that $a_n \neq 0$. We have

$$\begin{aligned} f(x) &= a_n x^n + a_{n+1}x^{n+1} + a_{n+2}x^{n+2} + \dots \\ &= a_n x^n \big(1 + (a_{n+1}/a_n)x + (a_{n+2}/a_n)x^2 + \dots\big) = a_n x^n\, g(x). \end{aligned}$$

The power series $g(x)$ is convergent with radius R. Theorem 6.40 implies g is continuous at 0, and clearly $g(0) = 1$. By Corollary 6.17, there is an open ball $B_\delta(0)$ on which $g(x) > 1/2$. On the deleted ball $B_\delta^\times(0)$, we have $|f(x)| > |a_n/2|\,|x|^n > 0$. ☐

An analytic function expanded in a power series centered at 0 can be easily separated into its even and odd parts (Example 5.16).

Proposition 6.47. *Let $R > 0$, and assume $f(x) = \sum_\ell a_\ell x^\ell$ converges absolutely for $|x| < R$. In the interval $(-R, R)$, the even and odd parts of f are given by the convergent power series*

$$\begin{aligned} f_{\text{even}}(x) &= a_0 + a_2x^2 + a_4x^4 + \dots = \sum_{k=0}^{\infty} a_{2k}x^{2k}, \\ f_{\text{odd}}(x) &= a_1 + a_3x^3 + a_5x^5 + \dots = \sum_{k=0}^{\infty} a_{2k+1}x^{2k+1}, \end{aligned}$$

the sums of the even-degree terms and odd-degree terms, respectively.

Proof. Substituting $-x$ for x, we have

$$f(-x) = \sum_{\ell=0}^{\infty} a_\ell(-x)^\ell = \sum_{\ell=0}^{\infty} (-1)^\ell a_\ell x^\ell = a_0 - a_1x + a_2x^2 - a_3x^3 + \dots.$$

Adding this to the series for $f(x)$, the odd-degree terms cancel and the even-degree terms are doubled. The even part of f is therefore the sum of the even-degree terms of the power series.

Similarly, subtracting the series for $f(-x)$ from the series for $f(x)$, the even-degree terms cancel and the odd-degree terms are doubled. The odd part of f is therefore the sum of the odd-degree terms of the power series. ☐

The Intermediate Value Theorem

Suppose f is a function and y is a real number. As we saw earlier when considering surjectivity of the squaring function $f(x) = x^2$, it may be a subtle issue to prove that the equation $f(x) = y$ has a real solution x.

The "Intermediate Value Theorem", Corollary 6.50 below, gives a useful sufficient condition under which an equation $f(x) = y$ is guaranteed to have a real solution.

Theorem 6.48. *Let $a < b$ be real numbers, and let $f : [a, b] \to \mathbf{R}$ be a continuous function such that $f(a) < 0 < f(b)$. There exists an x_0 in the open interval (a, b) such that $f(x_0) = 0$.*

Proof. Let $A = \{x \text{ in } [a, b] : f(x) < 0\}$ be the set of points at which $f(x)$ is negative. Since $f(a) < 0$, we have $a \in A$; consequently, A is a non-empty set of real numbers that is bounded above (by b). Define $x_0 = \sup A$. It suffices to prove $x_0 \in (a, b)$ and $f(x_0) = 0$.

By Corollary 6.17, if $f(x) < 0$, there exists an open interval centered at x on which f is negative, and if $f(x) > 0$, there exists an open interval centered at x on which f is positive. In particular, there exists a $\delta_1 > 0$ such that f is negative on $(a, a + \delta_1)$ (so $a < a + \delta_1 \leq x_0$), and there exists a $\delta_2 > 0$ such that f is positive on $(b - \delta_2, b)$ (so $x_0 \leq b - \delta_2 < b$). This completes the proof that $a < x_0 < b$.

By the trichotomy axiom, exactly one of the following alternatives holds: $f(x_0) < 0$, $f(x_0) > 0$, or $f(x_0) = 0$. It suffices to prove the first two are impossible.

If $f(x_0)$ were negative, Corollary 6.17 would imply there exists a $\delta > 0$ such that $f(x) < 0$ for $x_0 < x < x_0 + \delta$; this would mean x_0 is not an upper bound of A. Since $x_0 = \sup A$ *is* an upper bound of A, $f(x_0)$ is not negative.

If $f(x_0)$ were positive, Corollary 6.17 would imply there exists a $\delta > 0$ such that $f(x) > 0$ for $x_0 - \delta < x < x_0$; this would mean that $x_0 - \delta < x_0$ is an upper bound of A. Since $x_0 = \sup A$ is the *least* upper bound of A, $f(x_0)$ is not positive. □

Remark 6.49. The same conclusion holds if $f(b) < 0 < f(a)$: Apply the theorem to the continuous function $g(x) = -f(x)$, which satisfies the condition $g(a) < 0 < g(b)$.

Corollary 6.50 (The Intermediate Value Theorem)**.** *Let a and b be real numbers with $a < b$, and let $f : [a, b] \to \mathbf{R}$ be a continuous function* *such that* $f(a) \neq f(b)$. *If* y_0 *is a real number between* $f(a)$ *and* $f(b)$, *then there exists an* x_0 *in* (a, b) *such that* $f(x_0) = y_0$.

Proof. Apply Theorem 6.48 to the continuous function $g(x) = f(x) - y_0$, noting that $g(x_0) = 0$ if and only if $f(x_0) = y_0$. □

Remark 6.51. The Intermediate Value Theorem guarantees that the image of a closed, bounded interval $[a, b]$ under a continuous function f is itself an interval. (Theorem 6.57 below implies the image is actually a closed, bounded interval.)

Example 6.52. The rational function $f(x) = 1/x$, defined for $x \neq 0$, is continuous throughout its domain. Moreover, $f(-1) = -1 < 0$ and $f(1) = 1 > 0$. However, $f(x) \neq 0$ for all x. This does not contradict the Intermediate Value Theorem because the domain is not an interval. (In accord with the Intermediate Value Theorem, the sign of f does not change on any interval contained in the domain of f.)

Existence of Roots

Theorem 6.53. *Let* $c > 0$ *be real, and let* $n \geq 1$ *be an integer. There exists a unique positive real number* x_0 *such that* $x_0^n = c$.

Remark 6.54. A number x satisfying $x^n = c$ is called an *nth root* of c. The unique positive nth root of a positive real number c is customarily denoted $\sqrt[n]{c}$ or $c^{1/n}$.

The number $\sqrt{c} = c^{1/2}$ is called the *square root* of c. The number $\sqrt[3]{c} = c^{1/3}$ is called the *cube root* of c.

Proof. Let $f : \mathbf{R} \to \mathbf{R}$ be the polynomial function $f(x) = x^n$. We have $f(0) = 0 < c$, while by Theorem 2.26,

$$f(1 + c) = (1 + c)^n \geq 1 + nc \geq 1 + c > c.$$

The Intermediate Value Theorem applied to f on the interval $[0, 1 + c]$ guarantees there exists a real number x_0 such that $0 < x_0 < 1 + c$ and $x_0^n = c$.

There cannot be more than one such number: By Lemma 5.33, f is strictly increasing, hence injective. □

Theorem 6.55. *Let* n *be a positive integer. The* n*th root function* $f : [0, \infty) \to [0, \infty)$, *defined by* $f(x) = x^{1/n}$, *is bijective and continuous.*

Proof. The nth power function $g(x) = x^n$ is easily checked to be the inverse of f, so f is bijective. Since f is strictly increasing, f^{-1} is continuous on $(0, \infty)$ by Theorem 6.19.

To establish continuity at $x_0 = 0$, fix $\varepsilon > 0$, take $\delta = \varepsilon^n$, and note that if $0 \leq x < \delta$, then $0 \leq x^{1/n} < \varepsilon$. □

Remark 6.56. Let $x > 0$. We define *fractional exponents* by

$$x^{m/n} = (x^{1/n})^m = (x^m)^{1/n},$$

the unique positive real number whose nth power is x^m. It is straightforward to check that the laws of exponents in Theorem 2.25 hold for rational exponents.

The Extreme Value Theorem

Theorem 6.57. *Let $a < b$ be real numbers, and let $f : [a, b] \to \mathbf{R}$ be a continuous function. There exist points $x_{\min}$ and $x_{\max}$ in $[a, b]$ such that*

$$f(x_{\min}) \leq f(x) \leq f(x_{\max}) \quad \textit{for all } x \textit{ in } [a, b].$$

In words, a continuous function achieves a maximum value and a minimum value on every closed interval $[a, b]$ in its domain.

Proof. The strategy is first to show that a continuous function must be bounded, then to show the supremum and infimum are actually achieved.

(Continuous implies bounded). Assume contrapositively that f is unbounded on $[a, b]$. We will show there exists a point x_∞ at which f is discontinuous.

Since f is unbounded, for every natural number n, there exists a point x_n in $[a, b]$ such that $|f(x_n)| \geq n$. Choosing one such point for each natural number n gives a sequence (x_n) in $[a, b]$. The Bolzano-Weierstrass theorem implies there exists a subsequence $(x_{\nu(k)})$ that converges to some real number x_∞. By construction, $a \leq x_{\nu(k)} \leq b$ for all k, so $x_\infty \in [a, b]$ by Proposition 4.35. Finally,

$$|f(x_{\nu(k)})| \geq \nu(k) \geq k \quad \text{for each } k \geq 1,$$

which means the image sequence $y_k = f(x_{\nu(k)})$ is unbounded, and hence divergent. Since $(x_{\nu(k)}) \to x_\infty$ but the image sequence is not convergent, f is discontinuous at x_∞.

(Extrema are achieved). Since f is bounded, the image of f has a supremum β and an infimum α. It remains to prove there exist points $x_{\min}$ and $x_{\max}$ such that $f(x_{\min}) = \alpha$ and $f(x_{\max}) = \beta$.

For each positive integer n, $\beta - \frac{1}{n} < \beta$ is not an upper bound of the image, so there exists a point x_n such that $\beta - \frac{1}{n} < f(x_n) \leq \beta$. Again, we have constructed a sequence (x_n) in $[a, b]$, and by the Bolzano-Weierstrass theorem, there is a subsequence $(x_{\nu(k)})$ converging to some point $x_{\max}$ in $[a, b]$. Since

$$\beta - \tfrac{1}{\nu(k)} < f(x_{\nu(k)}) \leq \beta \quad \text{for all } k \geq 1,$$

continuity of f implies $f(x_{\max}) = \lim_k f(x_{\nu(k)}) = \beta$.

Achievement of α is proven entirely similarly. □

Example 6.58. The reciprocal function $f(x) = 1/x$, defined for all $x \neq 0$, is continuous throughout its domain, but is not bounded above or below. This does not contradict the Extreme Value Theorem, because the domain of f is not a closed, bounded interval.

Example 6.59. The polynomial function $f(x) = x$, defined on the interval $X = (0, 1)$, is bounded, but does not achieve a minimum or maximum value. There is no conflict with the Extreme Value Theorem because X is not a *closed* interval.

Example 6.60. The signum function (Example 6.3) achieves an absolute maximum of 1 at every positive real number, and achieves an absolute minimum of -1 at every negative real number. However, the domain $\mathbf{R} \setminus \{0\}$ is not a closed, bounded interval.

Infinite Limits

The symbols ∞ and $-\infty$ do not denote real numbers, but *extended* real numbers, which may be viewed as "endpoints" of the real number line.

Definition 6.61. Let X be unbounded above, and $f : X \to \mathbf{R}$ a function. An extended real number L is a *limit of f at* ∞ if for every sequence (x_k) in X such that $(x_k) \to \infty$, we have $\big(f(x_k)\big) \to L$. If this condition holds, we write $f(x) \to L$ as $x \to \infty$, or

$$L = \lim(f, \infty) = \lim_{x \to \infty} f(x).$$

Remark 6.62. A limit at infinity is unique (if it exists). Further, there is an analogous definition for the limit of f at $-\infty$ if the domain of f is unbounded below.

Theorems 4.31 and 4.41 immediately imply corresponding properties for functional limits:

Theorem 6.63. *Let X be unbounded above, and let f and g be functions on X.*

If $\lim(f, \infty)$ and $\lim(g, \infty)$ are finite, then

(i) $\lim(f \pm g, \infty) = \lim(f, \infty) \pm \lim(g, \infty)$.

(ii) $\lim(fg, \infty) = \lim(f, \infty) \cdot \lim(g, \infty)$.

(iii) *If $\lim(g, \infty) \neq 0$, then $\lim(f/g, \infty) = \lim(f, \infty)/\lim(g, \infty)$.*

If $\lim(f, \infty) = \infty$ and $\lim(g, \infty) = L$ is finite, then

(iv) $\lim(f + g, \infty) = \infty$.

(v) *If $\pm L > 0$, then $\lim(fg, \infty) = \pm\infty$.*

(vi) *If $f(x) \neq 0$ for all x in X, then $\lim(g/f, \infty) = 0$.*

Similar formulas hold for limits at $-\infty$.

Example 6.64. If $n \geq 1$ is an integer, then $x^n \to \infty$ as $x \to \infty$, and $x^n \to (-1)^n\infty$ as $x \to -\infty$.

Example 6.65. If p is a *monic* polynomial, i.e., if the degree is at least 1 and the leading coefficient is 1, then $p(x) \to \infty$ as $x \to \infty$, and $p(x) \to (-1)^n\infty$ as $x \to -\infty$. To prove this, write (for $x \neq 0$)

$$\begin{aligned} p(x) &= x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 \\ &= x^n(1 + a_{n-1}/x + \cdots + a_1/x^{n-1} + a_0/x^n). \end{aligned}$$

Inside the parentheses, each summand except the first approaches 0 as $|x| \to \infty$, so the terms in parentheses approach 1. The assertions follow from part (v) of the theorem.

Iteration, and Recursive Sequences

Definition 6.66. Let $f : X \to X$ be a function, so that f is composable with itself. We recursively define the (forward) *iterates* of f by

$$f^{[0]} = \iota, \qquad f^{[k+1]} = f \circ f^{[k]} \quad \text{for } k \geq 0.$$

That is, $f^{[1]} = f$, $f^{[2]} = f \circ f$, $f^{[3]} = f \circ f \circ f$, etc.

The pair (X, f) is called a *discrete dynamical system.*

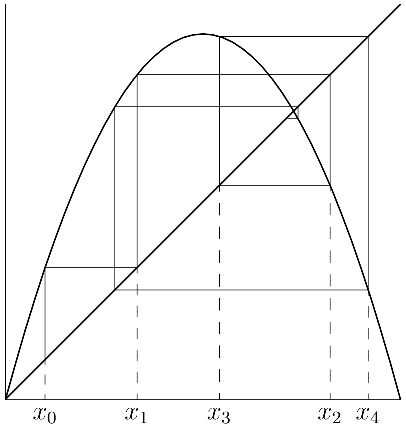
Example 6.67. If $f(x) = x^2 - 1$, then

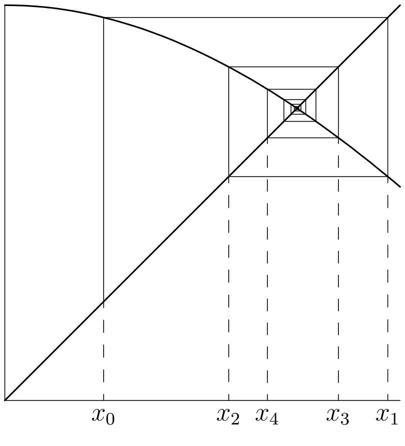
$$\begin{aligned}
f^{[2]}(x) &= f(x)^2 - 1 = (x^2 - 1)^2 - 1 = x^4 - 2x^2, \\
f^{[3]}(x) &= \big(f^{[2]}(x)\big)^2 - 1 = (x^4 - 2x^2)^2 - 1 = x^8 - 4x^6 + 4x^4 - 1, \\
f^{[4]}(x) &= \big(f^{[3]}(x)\big)^2 - 1 = (x^8 - 4x^6 + 4x^4 - 1)^2 - 1,
\end{aligned}$$

and so forth.

Definition 6.68. Let (X, f) be a discrete dynamical system. For each "seed" or "initial value" x_0 in X, there is a *recursive sequence* (x_k) defined by

$$x_{k+1} = f(x_k) \quad \text{for } k \geq 0.$$





Proposition 6.69. *Let X be non-empty, and let $f : X \to X$ be a* continuous *function. If a recursively-defined sequence converges to a point x_∞ in X, then $f(x_\infty) = x_\infty$.*

Proof. Since f is continuous,

$$f(x_\infty) = f\big(\lim_{k\to\infty} x_k\big) = \lim_{k\to\infty} f(x_k) = \lim_{k\to\infty} x_{k+1} = x_\infty. \qquad \square$$

Remark 6.70. A point x such that $f(x) = x$ is called a *fixed point* of f. The proposition says that if a recursive sequence converges, the limit is a fixed point of f.

Example 6.71 (Newton's method for square roots)**.** Let $c > 1$, and define the function $f : (0,\infty) \to (0,\infty)$ by

$$f(x) = \frac{1}{2}\left(x + \frac{c}{x}\right) = \frac{x^2 + c}{2x}.$$

Pick a seed x_0 with $x_0^2 > c$, such as $x_0 = c$. For $k \geq 0$, algebra gives

$$\begin{aligned} x_{k+1}^2 - c &= \frac{(x_k^2 + c)^2}{(2x_k)^2} - c \\ &= \frac{(x_k^4 + 2cx_k^2 + c^2) - 4cx_k^2}{(2x_k)^2} \\ &= \frac{x_k^4 - 2cx_k^2 + c^2}{(2x_k)^2} = \frac{(x_k^2 - c)^2}{(2x_k)^2}, \end{aligned}$$

which is positive by induction since $x_0^2 - c > 0$. Consequently,

$$x_k - x_{k+1} = x_k - \frac{1}{2}\left(x_k + \frac{c}{x_k}\right) = \frac{1}{2}\left(x_k - \frac{c}{x_k}\right) = \frac{x_k^2 - c}{2x_k} > 0.$$

We have shown that the recursive sequence (x_k) is positive (hence bounded below) and strictly decreasing, so (x_k) converges to a non-negative limit x_∞. By Proposition 6.69, we have

$$x_\infty = f(x_\infty) = \frac{x_\infty^2 + c}{2x_\infty}.$$

Rearranging, $x_\infty^2 = c$, so $x_\infty = \sqrt{c}$ since the limit is non-negative.

The convergence is remarkably rapid. Put

$$e_k = x_k^2 - c = (x_k + \sqrt{c})(x_k - \sqrt{c}) < 2c(x_k - \sqrt{c}),$$

essentially a constant times the amount by which x_k differs from $\sqrt{c}$. As shown above,

$$e_{k+1} = x_{k+1}^2 - c = \left(\frac{x_k^2 - c}{2x_k}\right)^2 = \frac{e_k^2}{4x_k^2} < \frac{e_k^2}{4c}.$$

That is, the *number of decimals of accuracy more than doubles* with each step of the iteration.

Exercises

Exercise 6.1. Let

$$\begin{aligned} p(x) &= a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n, \\ q(x) &= b_0 + b_1 x + b_2 x^2 + \cdots + b_m x^m, \end{aligned}$$

be polynomial functions with $a_n \neq 0$ and $b_m \neq 0$, and consider the rational function $f(x) = p(x)/q(x)$, defined on its natural domain. Prove:

(a) If $n < m$, then $f(x) \to 0$ as $|x| \to \infty$.

(b) If $n = m$, then $f(x) \to a_n/b_m$ as $|x| \to \infty$.

(c) If $n > m$, then $f(x) \to \operatorname{sgn}(a_n/b_m) \cdot \infty$ as $x \to \infty$.

Exercise 6.2. Let $p : \mathbf{R} \to \mathbf{R}$ be a monic polynomial, i.e.,

$$p(x) = x^n + a_{n-1}a^{n-1} + \cdots + a_1 x + a_0.$$

(a) Prove that if n is odd, then p has a real root.

(b) Prove that if n is even, then p has an absolute minimum.

Exercise 6.3. Supply a proof of statement (ii) in Theorem 6.29.

Exercise 6.4. For each real number x, we define the *floor* $\lfloor x \rfloor$ to be the largest integer n such that $n \leq x$. Similarly, the *ceiling* $\lceil x \rceil$ is the smallest integer n such that $x \leq n$.

(a) Sketch the graphs: $y = \lfloor x \rfloor$, $y = \lceil x \rceil$, $y = x - \lfloor x \rfloor$, $y = \lceil x \rceil - \lfloor x \rfloor$.

(b) Let $f(x) = \lceil x \rceil - \lfloor x \rfloor$. At which points x_0 does $\lim(f, x_0)$ exist? At which points is f continuous?

Exercise 6.5. Take $c = 3$ and $x_0 = 2$ in Newton's method. Find the next three terms of the sequence as reduced rational numbers. Using the fact that $1.7 < \sqrt{3} < 1.8$, find the smallest positive integer k such that $|x_k - \sqrt{3}| < 0.5 \times 10^{-10}$.

Exercise 6.6. Take $c = 5$ and $x_0 = 5/2$ in Newton's method. Find the next two terms of the sequence as reduced rational numbers. Using the fact that $2.2 < \sqrt{5} < 2.3$, find the smallest positive integer k such that $|x_k - \sqrt{5}| < 0.5 \times 10^{-12}$.

Exercise 6.7. Use Exercise 2.10 to prove $n^{1/n} = \sqrt[n]{n} \to 1$ as $n \to \infty$.

Exercise 6.8. Let $\rho > 0$. Prove that $\rho^{1/n} = \sqrt[n]{\rho} \to 1$ as $n \to \infty$. Suggestion: Reduce to the case $\rho > 1$, and use the preceding exercise.

Exercise 6.9. Let $\chi_{\mathbf{Q}}$ be Dirichlet's function,

$$\chi_{\mathbf{Q}}(x) = \begin{cases} 1 & \text{if } x \text{ is rational,} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

(a) Prove that $\chi_{\mathbf{Q}}$ is discontinuous at every real number.

(b) Let f be a continuous function. Prove that $f \cdot \chi_{\mathbf{Q}}$ is continuous at x if and only if $f(x) = 0$.

Exercise 6.10. Let f be Thomae's function,

$$f(x) = \begin{cases} 1/q & \text{if } x = p/q \text{ in lowest terms,} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

Prove that $\lim(f, x) = 0$ for all real x. (Thus f is continuous at every irrational number and discontinuous at every rational.)
Hint: If N is a positive integer, then $f(x) \geq 1/N$ if and only if $x \in \mathbf{Q}_N$. By Exercise 3.14 (b), every point of $\mathbf{Q}_N$ is isolated.

Exercise 6.11. Let I be an interval. A function $f : I \to I$ is said to be a *contraction* on I if there exists a positive real number $\mu < 1$ such that if x, y are elements of I, then $|f(x) - f(y)| \leq \mu|x - y|$.

(a) Prove that a contraction has at most one fixed point.

(b) Let x_0 be an element of I, and let (x_k) be the recursive sequence with seed x_0. Use induction to prove that $|x_{k+1} - x_k| \leq \mu^k |x_1 - x_0|$ for all $k \geq 0$.

(c) Prove that for all $k \geq 0$ and all $m > 0$,

$$|x_{k+m} - x_k| \leq |x_1 - x_0| \frac{\mu^k}{1 - \mu}.$$

Conclude that (x_k) is a Cauchy sequence.

(d) If the interval I is closed (but not necessarily bounded), then a contraction on I has a fixed point.

Exercise 6.12. Define $f : (0, \infty) \to (0, \infty)$ by $f(x) = x + (1/x)$.

(a) Prove that $|f(x) - f(y)| < |x - y|$ for all positive real numbers x and y. Is f a contraction on $(0, \infty)$?

(b) Let (x_k) be the recursive sequence defined by f and having seed $x_0 = 1$. Is (x_k) Cauchy?

Exercise 6.13. Fix a real number $b > 0$, and define $f(x) = \sqrt{b + x}$ for $x \geq 0$.

(a) There is a unique positive fixed point of f; find it (in terms of b).

(b) For which b is f a contraction on $(0, \infty)$?
Suggestion: Multiply and divide by the conjugate expression.

(c) Let $x_0 > 0$ be the seed for the recursive sequence (x_k) generated by f. Show that (x_k) converges to the fixed point (whether or not f is a contraction).

(d) Evaluate $\sqrt{1 + \sqrt{1 + \sqrt{1 + \dots}}}$ and $\sqrt{2 + \sqrt{2 + \sqrt{2 + \dots}}}$.

Exercise 6.14. Let $f(x) = 1/(1 + x)$ for $x \geq 0$.

(a) If $0 < \delta < 1$, then f is a contraction on $[\delta, \infty)$.

(b) Evaluate the continued fraction $\cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \ddots}}}$.

Exercise 6.15. Let X be non-empty. A subset O of X is *relatively open* (in X) if there exists an open subset U of $\mathbf{R}$ such that $O = X \cap U$. For example, $O = [0, 1)$ is relatively open in $X = [0, 2]$.

(a) Prove that $\varnothing$ and X are relatively open in X.

(b) Prove that if $\{O_i\}_{i \in I}$ is an arbitrary family of relatively open subsets of X, then $\bigcup_i O_i$ is relatively open in X.

(c) Prove that if $\{O_i\}_{i=1}^{n}$ is a finite family of relatively open subsets of X, then $\bigcap_i O_i$ is relatively open in X.

(d) Suppose $f : X \to \mathbf{R}$ is a function. Prove that f is continuous on X if and only if the following condition holds:

> For every open set V of real numbers, the preimage $f^{-1}(V)$ is relatively open in X.

Exercise 6.16. Define the *Heaviside function* or *unit step function* by

$$H(x) = \begin{cases} 0 & \text{if } x < 0, \\ 1 & \text{if } x \geq 0. \end{cases}$$

Let $(a_k)_{k=1}^{\infty}$ be a sequence that injectively enumerates the rational numbers (i.e., assume every rational number appears *exactly once* as a term of the sequence), and define

$$f(x) = \sum_{k=1}^{\infty} \frac{H(x - a_k)}{2^k}.$$

(a) Prove the series $f(x)$ converges absolutely for every real x, and the sum is between 0 and 1.

(b) Show that for every real number x, the one-sided limit $\lim(f, x^-)$ is the sum of $\frac{1}{2^k}$ taken over natural numbers k such that $a_k < x$, and find a similar description for the right-hand limit $\lim(f, x^+)$.

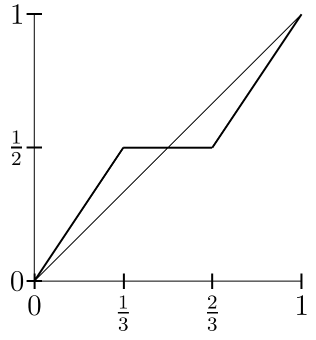
(c) Prove that the function f is strictly increasing, is discontinuous at every rational number, and is continuous at every irrational.

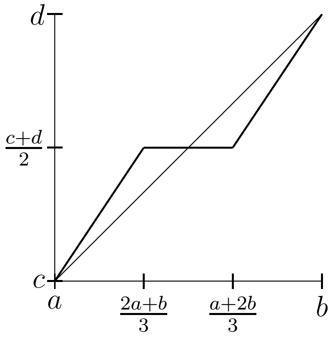
Exercise 6.17. This exercise sketches the construction of the *Cantor function*, a continuous, non-decreasing surjection $f : [0, 1] \to [0, 1]$ that is constant on every open interval in the complement of the Cantor set.

(a) Find a piecewise description of the function S in the left-hand diagram below. The graph is horizontal on the middle third, passes through the lower-left and upper-right corners, and consists of three segments.

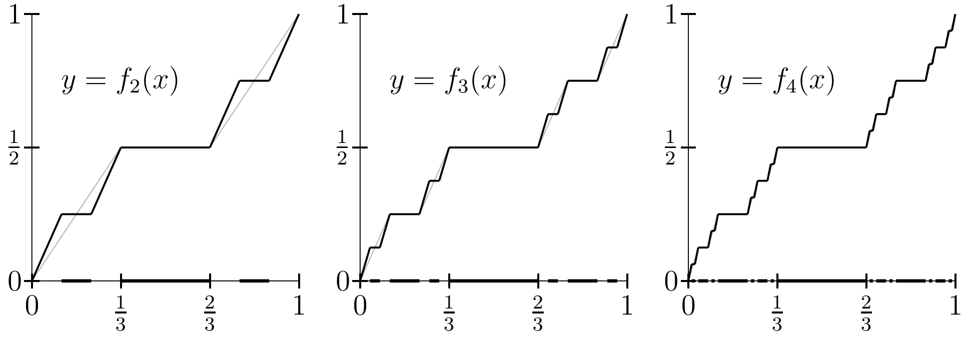
Let $a < b$ and $c < d$ be real. Use shifting and scaling to express the function T in the right-hand diagram in terms of S.

(b) Prove that in the right-hand diagram, the maximum vertical distance between the diagonal and the graph of T is no larger than $\frac{1}{2}(d - c)$. (The actual maximum separation is $\frac{1}{6}(d - c)$, but the cruder estimate suffices for our needs.)





(c) Recursively construct functions on $[0, 1]$ as follows. Let $f_0(x) = x$ and $f_1(x) = S(x)$. To obtain f_{n+1} from f_n, separately consider each interval on which f_n is non-constant and "replace $f_n(x)$ by $T(x)$" on that interval:



Use induction to prove that on each interval where f_n is non-constant, i.e., on each component of K_n, the function changes by $\frac{1}{2^n}$.

Use part (b) to conclude that $\sup |f_n(x) - f_{n+1}(x)| < \frac{1}{2^{n+1}}$. (The supremum is taken over all x in $[0, 1]$.)

Use the triangle inequality and the geometric series formula to prove $\sup |f_n(x) - f_{n+k}(x)| < \frac{1}{2^n}$ for all $k \geq 0$.

(d) By part (c), for each x in $[0, 1]$, the sequence $\big(f_n(x)\big)$ is Cauchy. Let $f(x)$ be the limit. Prove that f is non-decreasing, takes values in $[0, 1]$, is constant on each interval in the complement of the Cantor set, and is continuous.

Chapter 7

Integration

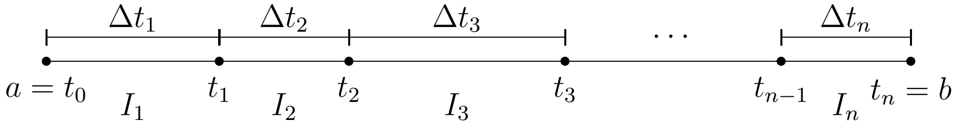
7.1 Partitions and Sums

Definition 7.1. Let $a < b$ be real numbers. A *partition* of the closed interval $I = [a, b]$ is a finite collection of points $P = \{t_i\}_{i=0}^{n}$ such that

$$a = t_0 < t_1 < t_2 < \cdots < t_n = b.$$

The interval $I_i = [t_{i-1}, t_i]$ is called the *ith subinterval* of the partition. Its *length* is

$$\Delta t_i = t_i - t_{i-1}.$$



Definition 7.2. Let P and Q be partitions of I. We say Q is a *refinement* of P if $P \subseteq Q$.

Remark 7.3. A refinement of P is obtained by adding finitely many points to P. If P and P' are partitions of I, their union $Q = P \cup P'$ is a "common refinement", i.e., a refinement of each.

Example 7.4. Let $n \geq 1$ be an integer. An *equal-length partition* has n subintervals, each of length $\Delta t = (b - a)/n$:

$$P_n = \{a, a + \Delta t, a + 2\Delta t, \ldots, a + n\Delta t = b\}.$$

In this example, $P_{n'}$ is a refinement of P_n if and only if n divides n'.

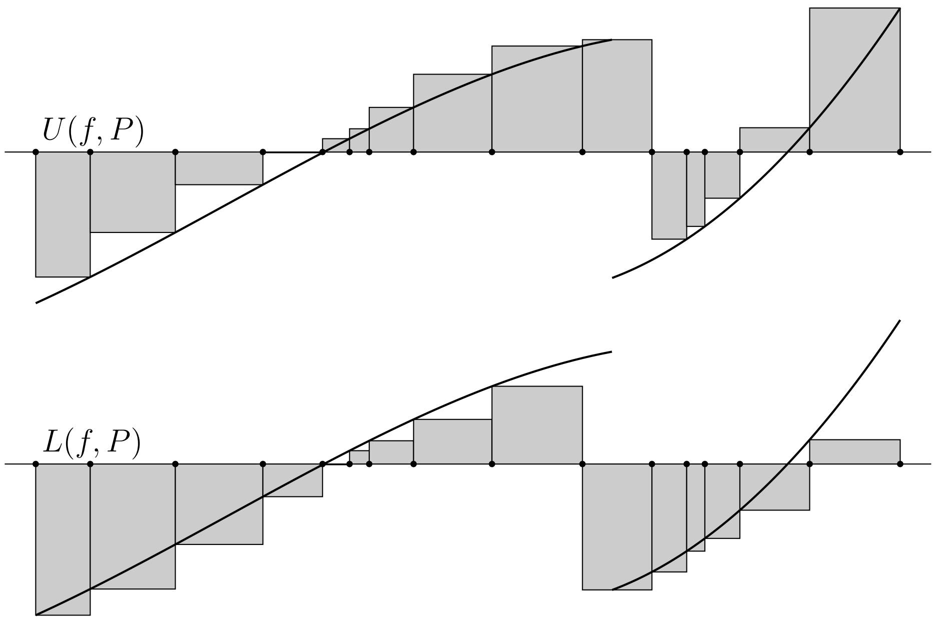
Definition 7.5. Let $f : [a, b] \to \mathbf{R}$ be a *bounded* function. For each partition P of $I = [a, b]$ into n subintervals, and for each index i with $1 \le i \le n$, put

$$m_i = \inf\{f(t) : t \in I_i\}, \qquad M_i = \sup\{f(t) : t \in I_i\}.$$

We define the *lower sum* and *upper sum* of f with respect to P by

$$L(f, P) = \sum_{i=1}^{n} m_i\,\Delta t_i, \qquad U(f, P) = \sum_{i=1}^{n} M_i\,\Delta t_i.$$

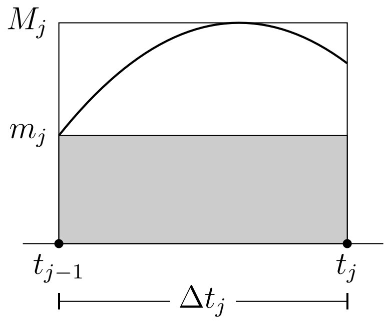
Remark 7.6. Geometrically, the summand $M_i\,\Delta t_i$ is the signed area of the lowest rectangle that lies entirely above the graph of f on I_i, and the summand $m_i\,\Delta t_i$ is the signed area of the highest rectangle that lies entirely below the graph of f on I_i.

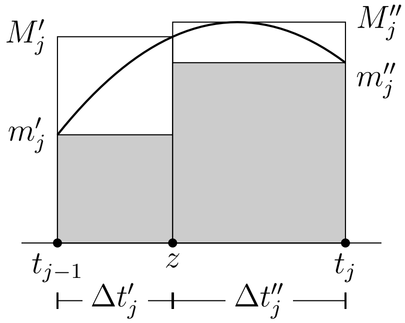


Lemma 7.7. *Let $f : [a, b] \to \mathbf{R}$ be a bounded function. If P and Q are partitions of $[a, b]$ and Q is a refinement of P, i.e., $P \subseteq Q$, then*

$$L(f, P) \le L(f, Q) \le U(f, Q) \le U(f, P).$$

Proof. Since every refinement of P is obtained by appending finitely many points to P, induction on the number of points in $Q \setminus P$ reduces the claim to the case where $Q = P \cup \{z\}$ for some z.





Assume $z \in I_j$ for definiteness, and write $I_j = [t_{j-1}, z] \cup [z, t_j]$. Let m_j' and m_j'' denote the respective infima of f on the subintervals $I_j' = [t_{j-1}, z]$ and $I_j'' = [z, t_j]$. Since each subinterval is contained in I_j, we have $m_j \le m_j'$ and $m_j \le m_j''$. Consequently,

$$m_j \, \Delta t_j = m_j \, (\Delta t_j' + \Delta t_j'') \le m_j' \, \Delta t_j' + m_j'' \, \Delta t_j''.$$

Since the lower sums $L(f, P)$ and $L(f, Q)$ have identical summands if $i \ne j$, we have $L(f, P) \le L(f, Q)$.

A completely analogous argument shows $U(f, Q) \le U(f, P)$. ☐

Proposition 7.8. *Let* $f : [a, b] \to \mathbf{R}$ *be a bounded function. If* P *and* P' *are arbitrary partitions of* $[a, b]$*, then* $L(f, P) \le U(f, P')$*.*

Proof. The partition $Q = P \cup P'$ is a refinement of both P and P'. By Lemma 7.7, $L(f, P) \le L(f, Q) \le U(f, Q) \le U(f, P')$. ☐

Remark 7.9. Let $\mathscr{L} = \{L(f, P) : P \text{ a partition of } [a, b]\}$ be the set of all lower sums of f, and let $\mathscr{U} = \{U(f, P) : P \text{ a partition of } [a, b]\}$ be the set of all upper sums of f. Proposition 7.8 guarantees that every lower sum of f is bounded above by every upper sum. In particular, the set $\mathscr{L}$ is non-empty and bounded above, and $\mathscr{U}$ is non-empty and bounded below.

Definition 7.10. Let $f : [a, b] \to \mathbf{R}$ be bounded. The *lower integral* of f on $[a, b]$ is the real number $L(f, [a, b]) = \sup \mathscr{L}$, and the *upper integral* of f on $[a, b]$ is the real number $U(f, [a, b]) = \inf \mathscr{U}$.

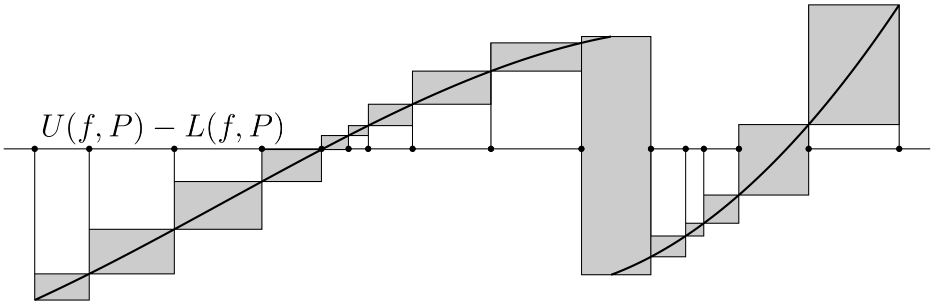
If $L(f, [a, b]) = U(f, [a, b])$, we say f is *integrable* on $[a, b]$, and call

$$L(f, [a, b]) = U(f, [a, b]) = \int_a^b f = \int_a^b f(t) \, dt$$

the *integral* of f over $[a, b]$.

Proposition 7.11. *A bounded function $f : [a,b] \to \mathbf{R}$ is integrable on $[a,b]$ if and only if the following condition holds:*

> *For every $\varepsilon > 0$, there exists a partition P of $[a,b]$ such that $U(f,P) - L(f,P) < \varepsilon$.*



Proof. Suppose f is integrable, and let $\varepsilon > 0$ be arbitrary. By definition of supremum and infimum there exist partitions P' and P'' such that

$$U(f,P'') - \left(\int_a^b f\right) < \varepsilon/2, \qquad \left(\int_a^b f\right) - L(f,P') < \varepsilon/2.$$

If $P = P' \cup P''$, then $L(f,P') \le L(f,P) \le U(f,P) \le U(f,P'')$, so

$$U(f,P) - L(f,P) \le U(f,P'') - L(f,P') < \varepsilon.$$

Inversely, suppose f is *not* integrable. If we set $\varepsilon = \inf \mathscr{U} - \sup \mathscr{L}$, then $U(f,P) - L(f,P) \ge \varepsilon > 0$ for every partition P. $\square$

Example 7.12. Let f be a constant function, say $f(t) = c$ for all t. For every partition of $[a,b]$ and for every subinterval of the partition, $m_i = c = M_i$. Consequently, every lower sum and every upper sum is equal to $c(b-a)$, so

$$\int_a^b f = \int_a^b c\,dt = c(b-a).$$

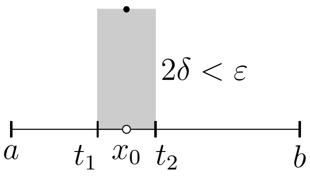
Example 7.13. Let $a < x_0 < b$ be real numbers, and let $f = \chi_{\{x_0\}}$, i.e., $f(t) = 1$ if $t = x_0$ and $f(t) = 0$ otherwise. We will show the function f is integrable on $[a,b]$, and the integral is equal to 0.

Fix $\varepsilon > 0$. We seek a partition P of the form

$$t_0 = a, \quad t_1 = x_0 - \delta, \quad t_2 = x_0 + \delta, \quad t_3 = b$$

for which $U(f,P) - L(f,P) < \varepsilon$. It suffices to arrange that $a < x_0 - \delta$, $x_0 + \delta < b$, and $2\delta < \varepsilon$: Under these conditions, we have

i	m_i	M_i	Δt_i	$(M_i - m_i)\,\Delta t_i$,
1	0	0	$(x_0 - \delta) - a$	0,
2	0	1	2δ	2δ,
3	0	0	$b - (x_0 + \delta)$	0,



so that $U(f,P) - L(f,P) = 2\delta < \varepsilon$. Since $\varepsilon > 0$ was arbitrary, f is integrable on $[a,b]$. Since every lower sum is 0, the integral of f is 0.

Obvious modifications of this argument handle the possibility that $x_0 = a$ or $x_0 = b$.

Example 7.14. Let $f = \chi_{\mathbf{Q}}$ be the characteristic function of $\mathbf{Q}$. Then f is *not* integrable on $[a,b]$, no matter how $a < b$ are chosen. Indeed, let P be a partition of $[a,b]$. In each subinterval, there exist both rational numbers and irrational numbers (see Theorem 3.81 and subsequent remarks), so f takes on the values 0 and 1 in *every* subinterval, i.e., $m_i = 0$ and $M_i = 1$ for every i. Thus,

$$L(f,P) = \sum_{i=1}^{n} 0\,\Delta t_i = 0, \qquad U(f,P) = \sum_{i=1}^{n} 1\,\Delta t_i = (b-a),$$

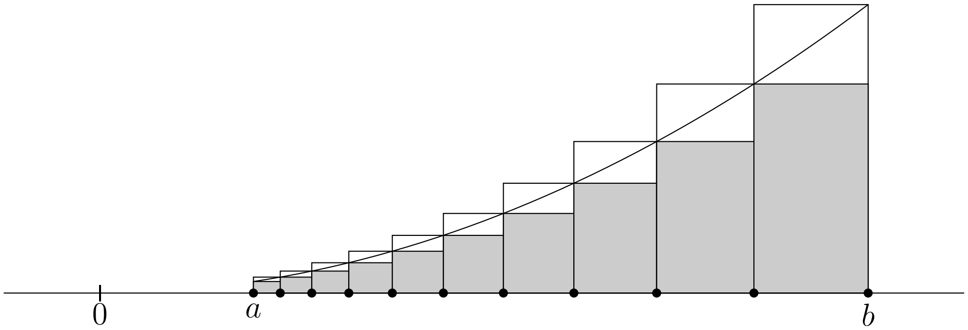
regardless of P. This means $0 = L(f,[a,b]) < U(f,[a,b]) = b - a$, so f is not integrable on $[a,b]$.

Example 7.15. Assume $0 < a < b$, and let k be a positive integer. Consider the monomial function $f(t) = t^k$ on $[a,b]$. We will prove *from the definition* that f is integrable on $[a,b]$, and

$$\int_a^b t^k\,dt = \frac{b^{k+1} - a^{k+1}}{k+1}.$$

It is convenient to use a “geometric” partition, for which the ratio of the lengths of consecutive intervals is constant, see diagram below.

Let $n > 0$ be the number of subintervals, and put $\rho = \sqrt[n]{b/a} > 1$, so that $b = a\rho^n$. The partition is $P = \{t_i\}_{i=0}^n$, with $t_i = a\rho^i$, so we have $\Delta t_i = a\rho^i - a\rho^{i-1} = a(\rho - 1)\rho^{i-1}$ for $i = 1, \dots, n$. Since f is increasing on $[a,b]$, $m_i = f(t_{i-1}) = (a\rho^{i-1})^k$ and $M_i = f(t_i) = (a\rho^i)^k$. Because $M_i = \rho^k m_i$ for each i, the upper sum is ρ^k times the lower sum.



The general term of the lower sum is

$$f(t_{i-1})\,\Delta t_i = (a\rho^{i-1})^k\,a(\rho-1)\rho^{i-1} = a^{k+1}(\rho-1)\cdot(\rho^{k+1})^{i-1}.$$

The geometric sum formula $\displaystyle\sum_{i=1}^{n} r^{i-1} = \sum_{i=0}^{n-1} r^i = \frac{r^n-1}{r-1}$ gives

$$\begin{aligned} L(f,P) = \sum_{i=1}^{n} f(t_{i-1})\,\Delta t_i &= a^{k+1}(\rho-1)\cdot\sum_{i=1}^{n}(\rho^{k+1})^{i-1} \\ &= a^{k+1}(\rho-1)\cdot\frac{(\rho^{k+1})^n-1}{\rho^{k+1}-1} \\ &= a^{k+1}(\rho-1)\cdot\frac{(\rho^n)^{k+1}-1}{\rho^{k+1}-1} \\ &= a^{k+1}\big((\rho^n)^{k+1}-1\big)\cdot\frac{\rho-1}{\rho^{k+1}-1}. \end{aligned}$$

To simplify, recall that $\rho^n = b/a$, so

$$a^{k+1}\big((\rho^n)^{k+1}-1\big) = a^{k+1}\big((b/a)^{k+1}-1\big) = b^{k+1}-a^{k+1}.$$

Note further that

$$\frac{\rho^{k+1}-1}{\rho-1} = \sum_{i=0}^{k}\rho^i = 1+\rho+\rho^2+\cdots+\rho^k.$$

Call this sum $S(\rho)$ for brevity. By Exercise 6.8, the ratio $\rho = (b/a)^{1/n}$ approaches 1 as $n\to\infty$. Consequently, the polynomial $S(\rho)$ approaches $\lim(S,1) = k+1$, and thus

$$L(f,P) = (b^{k+1}-a^{k+1})\,\frac{\rho-1}{\rho^{k+1}-1} = \frac{b^{k+1}-a^{k+1}}{S(\rho)} \longrightarrow \frac{b^{k+1}-a^{k+1}}{k+1}.$$

Since $L(f, P) \leq \sup \mathscr{L} = L(f, [a, b])$ for every P, i.e., the lower integral is no smaller than any particular lower sum,

$$\frac{b^{k+1} - a^{k+1}}{k+1} = \lim_{n \to \infty} \frac{b^{k+1} - a^{k+1}}{S(\rho)} \leq L(f, [a, b]).$$

As noted earlier, $U(f, P) = \rho^k \cdot L(f, P)$, so

$$U(f, [a, b]) \leq \lim_{n \to \infty} U(f, P) = \lim_{n \to \infty} \rho^k \cdot L(f, P) = \frac{b^{k+1} - a^{k+1}}{k+1}.$$

The lower and upper integrals therefore have the same value; this simultaneously proves that f is integrable on $[a, b]$, and evaluates the integral.

7.2 Properties of the Integral

The result of Example 7.15 is fundamental, but the intricacy of the process should convince you that calculation of integrals from the definition should be avoided when possible. In order to calculate integrals from more theoretical considerations, we establish several general properties, collected here for convenience:

Theorem 7.16. *Let $a < b$ be real numbers, and f, g functions on $[a, b]$.*

(i) (*Linearity*) *If f and g are integrable on $[a, b]$, and if k is a real number, then the functions $f \pm g$ and kf are integrable on $[a, b]$, and*

$$\int_a^b (f + g) = \int_a^b f + \int_a^b g, \qquad \int_a^b (kf) = k \int_a^b f.$$

(ii) (*Monotonicity*) *If f and g are integrable on $[a, b]$, and if $f(t) \leq g(t)$ for all t in $[a, b]$, then*

$$\int_a^b f \leq \int_a^b g.$$

(iii) (*Triangle inequality*) *If f is integrable on $[a, b]$, then $|f|$ is integrable on $[a, b]$, and*

$$\left| \int_a^b f \right| \leq \int_a^b |f|.$$

(iv) (*Patching property*) *If c is a real number with $a < c < b$, then f is integrable on $[a, b]$ if and only if f is integrable on the two intervals $[a, c]$ and $[c, b]$, and in this event*

$$\int_a^b f = \int_a^c f + \int_c^b f.$$

Remark 7.17. Theorem 7.16 grants computational license to integrate sums and differences "term by term", and to factor constants out of an integral; to integrate a (non-strict) inequality, preserving the inequality; to move an absolute value inside an integral at the expense of making the expression larger; and to break an interval of integration into two (or finitely many) pieces, and to add up the resulting sub-integrals.

Proof. (Linearity). Let $P = \{t_i\}_{i=0}^n$ be an arbitrary partition of $[a, b]$. For each $i = 1, \dots, n$, let $I_i = [t_{i-1}, t_i]$, and set $m_i^f = \inf\{f(t) : t \in I_i\}$, $m_i^g = \inf\{g(t) : t \in I_i\}$, $m_i^{f+g} = \inf\{(f + g)(t) : t \in I_i\}$. Since

$$\{f(t) + g(t) : t \in I_i\} \subseteq \{f(s) + g(t) : s, t \in I_i\}$$

(a more restrictive predicate defines a smaller set), we have

$$\begin{aligned} m_i^f + m_i^g &= \inf\{f(s) : s \in I_i\} + \inf\{g(t) : t \in I_i\} \\ &= \inf\{f(s) + g(t) : s, t \in I_i\} \\ &\leq \inf\{f(t) + g(t) : t \in I_i\} = m_i^{f+g}. \end{aligned}$$

(Loosely, we expect equality when the infima of f and g are achieved at the same point of I_i.) With analogous notation for the suprema, $M_i^{f+g} \leq M_i^f + M_i^g$. Adding up these inequalities,

$$L(f, P) + L(g, P) \leq L(f + g, P) \leq U(f + g, P) \leq U(f, P) + U(g, P)$$

for every partition P. Taking suprema over P for the lower sums and infima over P for the upper sums,

$$\int_a^b f + \int_a^b g \leq L(f + g, [a, b]) \leq U(f + g, [a, b]) \leq \int_a^b f + \int_a^b g.$$

This shows simultaneously that $f + g$ is integrable on $[a, b]$, and that the integral has the stated value.

Integrability of $f - g = f + (-1)g$ will follow once we prove kg is integrable for each real number k. To establish the latter claim, we consider two cases: $k \geq 0$, and $k = -1$.

Let k be a real number. With notation as above, if P is an arbitrary partition of $[a, b]$ and $i = 1, \dots, n$ is an index, then

$$\{(kf)(t) : t \in I_i\} = k\{f(t) : t \in I_i\};$$

that is, the set of values of kf on I_i is k multiplied by the set of values of f (see Definition 3.24). If $k \geq 0$, we have

$$\begin{aligned} m_i^{kf} &= \inf\{(kf)(t) : t \in I_i\} = k \inf\{f(t) : t \in I_i\} = km_i^f, \\ M_i^{kf} &= \sup\{(kf)(t) : t \in I_i\} = k \sup\{f(t) : t \in I_i\} = kM_i^f \end{aligned}$$

by Proposition 3.49 and Exercise 3.8. Forming lower and upper sums,

$$\begin{aligned} L(kf, P) &= \sum_{i=1}^n m_i^{kf}\,\Delta t_i = k \sum_{i=1}^n m_i^f\,\Delta t_i = kL(f, P), \\ U(kf, P) &= \sum_{i=1}^n M_i^{kf}\,\Delta t_i = k \sum_{i=1}^n M_i^f\,\Delta t_i = kU(f, P). \end{aligned}$$

Since f is integrable on $[a, b]$,

$$\begin{aligned} L(kf, [a, b]) &= \sup_P L(kf, P) = \sup_P kL(f, P) \\ &= k \sup_P L(f, P) = k \inf_P U(f, P) \\ &= \inf_P kU(f, P) = \inf_P U(kf, P) = U(kf, [a, b]), \end{aligned}$$

proving that kf is integrable on $[a, b]$, and the value of the integral is k times the integral of f.

If instead $k = -1$, then

$$\begin{aligned} m_i^{-f} &= \inf\{(-f)(t) : t \in I_i\} = -\sup\{f(t) : t \in I_i\} = -M_i^f, \\ M_i^{-f} &= \sup\{(-f)(t) : t \in I_i\} = -\inf\{f(t) : t \in I_i\} = -m_i^f. \end{aligned}$$

Forming lower and upper sums gives

$$\begin{aligned} L(-f, P) &= \sum_{i=1}^n m_i^{-f}\,\Delta t_i = -\sum_{i=1}^n M_i^f\,\Delta t_i = -U(f, P), \\ U(-f, P) &= \sum_{i=1}^n M_i^{-f}\,\Delta t_i = -\sum_{i=1}^n m_i^f\,\Delta t_i = -L(f, P). \end{aligned}$$

Taking the supremum of the lower sums and the infimum of the upper sums,

$$L(-f,[a,b]) = -U(f,[a,b]) = -L(f,[a,b]) = U(-f,[a,b]),$$

proving that $-f$ is integrable on $[a,b]$, and the integral of $-f$ is minus the integral of f.

(Monotonicity). Left to you, see Exercise 7.1.

(Triangle inequality). Let f be integrable on $[a,b]$. Fix $\varepsilon > 0$ and choose a partition P of $[a,b]$ such that $U(f,P) - L(f,P) < \varepsilon$.

By the reverse triangle inequality,

$$|f(x)| - |f(y)| \leq |f(x) - f(y)| \quad \text{for all } x,\ y \text{ in } [a,b].$$

In particular, letting I_i be the ith subinterval of P and taking the supremum over x in I_i, then taking the infimum over y in I_i, we have

$$M_i^{|f|} - m_i^{|f|} \leq |M_i^f - m_i^f| = M_i^f - m_i^f.$$

Multiplying by Δt_i and summing over i,

$$U(|f|,P) - L(|f|,P) \leq U(f,P) - L(f,P) < \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, $|f|$ is integrable on $[a,b]$.

Now, $-|f(t)| \leq f(t) \leq |f(t)|$ for all t in $[a,b]$. Monotonicity gives

$$-\int_a^b |f| \leq \int_a^b f \leq \int_a^b |f|, \quad \text{or} \quad \left|\int_a^b f\right| \leq \int_a^b |f|.$$

(Patching property). Let $a < c < b$ be real numbers, $f : [a,b] \to \mathbf{R}$ a function, and fix $\varepsilon > 0$ arbitrarily.

Suppose f is integrable on $[a,b]$. Choose a partition P of $[a,b]$ such that

$$\sum_{i=1}^n (M_i^f - m_i^f)\,\Delta t_i = U(f,P) - L(f,P) < \varepsilon.$$

Replacing P with $P \cup \{c\}$, we may assume $c \in P$ while preserving the preceding inequality.

The set $P' = [a,c] \cap P$ is a partition of $[a,c]$. Since every subinterval of P' is a subinterval of P, and every summand in $U(f,P) - L(f,P)$ is non-negative, $U(f,P') - L(f,P') < \varepsilon$. By Proposition 7.11, f is integrable on $[a,c]$. A similar argument shows f is integrable on $[c,b]$.

Conversely, suppose f is integrable on $[a, c]$ and on $[c, b]$. Choose partitions P' and P'' for $[a, c]$ and $[c, b]$ such that

$$U(f, P') - L(f, P') < \varepsilon/2, \qquad U(f, P'') - L(f, P'') < \varepsilon/2.$$

The union $P = P' \cup P''$ is a partition of $[a, b]$ for which

$$U(f, P) - L(f, P) = \big(U(f, P') - L(f, P')\big) + \big(U(f, P'') - L(f, P'')\big) < \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, f is integrable on $[a, b]$.

In either case, $L(f, P) = L(f, P') + L(f, P'')$, which proves

$$\int_a^b f = \int_a^c f + \int_c^b f.$$

This completes the proof of Theorem 7.16. □

Corollary 7.18. *Let f and g be functions on some interval $[a, b]$, and assume $f(t) = g(t)$ except at finitely many points. If f is integrable on $[a, b]$, then g is integrable on $[a, b]$, and the integral of g is equal to the integral of f.*

Proof. Consider the function $h = g - f$, which by hypothesis is non-zero at only finitely many points. That is, there exist points $x_1, x_2, \dots, x_m$ in $[a, b]$ and real numbers $k_1, k_2, \dots, k_m$ such that

$$h = \sum_{j=1}^{m} k_j \, \chi_{\{x_j\}}$$

By Example 7.13, each function $\chi_{\{x_j\}}$ is integrable and has integral equal to 0. By linearity of the integral, h itself is integrable, and

$$\int_a^b h = \sum_{j=1}^{m} k_j \int_a^b \chi_{\{x_j\}} = 0.$$

But $g = f + h$, so linearity implies g is integrable, and

$$\int_a^b g = \int_a^b (f + h) = \int_a^b f + \int_a^b h = \int_a^b f. \qquad \square$$

Corollary 7.19. *If $f : [a, b] \to \mathbf{R}$ is integrable, and if $m \le f(t) \le M$ for all but finitely many t in $[a, b]$, then*

$$m(b - a) \le \int_a^b f \le M(b - a).$$

Proof. This follows immediately from monotonicity and the preceding corollary. ☐

The patching property can be expressed in a symmetric form by making two conventions.

Definition 7.20. If $f : [a, b] \to \mathbf{R}$ is integrable, we define

$$\int_a^a f = 0, \quad \int_b^a f = -\int_a^b f.$$

When working with domains of integration, we continue to use "closed interval" notation, and write "$[b, a]$" even if $a < b$, interpreting the integral from b to a as indicated.

Corollary 7.21 (The cocycle property)**.** *Let I be an interval of real numbers, and let $f : I \to \mathbf{R}$ be integrable. For all a, b, c in I,*

$$\int_a^b f + \int_b^c f + \int_c^a f = 0.$$

Proof. First assume $a \leq c \leq b$. By the patching property and our conventions on swapping limits of integration,

$$\int_a^b f = \int_a^c f + \int_c^b f = -\left(\int_b^c f + \int_c^a f\right),$$

so the cocycle property holds in this case. But the stated equality is invariant under permutation of the symbols a, b, and c, and is therefore true regardless of the ordering of the numbers a, b, and c. ☐

A Transformation Theorem

Theorem 7.22. *Let $I = [a, b]$ be a domain of integration, and assume f is integrable on I.*

(i) *For every real number c,*

$$\int_{a+c}^{b+c} f(t - c)\, dt = \int_a^b f(t)\, dt.$$

(ii) *If $\mu \neq 0$ is real, then*

$$\mu \int_{a/\mu}^{b/\mu} f(\mu t)\, dt = \int_a^b f(t)\, dt.$$

Proof. (i) Define $\tau : I + c \to I$ by $\tau(t) = t - c$. For each partition $P = \{t_i\}_{i=0}^n$ of $[a + c, b + c]$, there is a unique partition $P' = \{t'_i\}_{i=0}^n$ of $[a, b]$ defined by $t'_i = \tau(t_i) = t_i - c$.

Since $f \circ \tau$ achieves precisely the same values in I_i that f achieves in I'_i we have

$$m_i'^f = m_i^{f\circ\tau}, \qquad M_i'^f = M_i^{f\circ\tau}.$$

Further, $\Delta t'_i = (t_i - c) - (t_{i-1} - c) = \Delta t_i$. Consequently,

$$L(f, P') = \sum_{i=1}^n m_i'^f \,\Delta t'_i = \sum_{i=1}^n m_i^{f\circ\tau}\,\Delta t_i = L(f \circ \tau, P),$$

and similarly $U(f, P') = U(f \circ \tau, P)$. The theorem follows at once by taking the supremum of the lower sums and the infimum of the upper sums.

(ii) Define $\tau : I/\mu = [a/\mu, b/\mu] \to I$ by $\tau(t) = \mu t$. The preceding proof goes through with straightforward modifications, notably that $\Delta t'_i = \mu\,\Delta t_i$. □

Corollary 7.23. *Let $a > 0$, and assume f is integrable on $[-a, a]$.*

(i) *If f is even, i.e., if $f(-t) = f(t)$ for all t, then*

$$\int_{-a}^a f(t)\,dt = 2\int_0^a f(t)\,dt.$$

(ii) *If f is odd, i.e., if $f(-t) = -f(t)$ for all t, then*

$$\int_{-a}^a f(t)\,dt = 0.$$

Proof. By part (ii) of the preceding theorem, with $\mu = -1$,

$$\int_{-a}^0 f(t)\,dt = -\int_a^0 f(-t)\,dt = \int_0^a f(-t)\,dt.$$

Breaking up the integral at 0,

$$\begin{aligned}\int_{-a}^a f(t)\,dt &= \int_{-a}^0 f(t)\,dt + \int_0^a f(t)\,dt\\ &= \int_0^a f(-t)\,dt + \int_0^a f(t)\,dt = \int_0^a \big(f(-t) + f(t)\big)\,dt.\end{aligned}$$

Both parts follow immediately. □

7.3 Criteria for Integrability

Proposition 7.24. *If $f : [a, b] \to \mathbf{R}$ is non-decreasing, then f is integrable on $[a, b]$.*

Proof. Since f is non-decreasing, $f(a) \leq f(t) \leq f(b)$ for all t in $[a, b]$. In particular, f is bounded.

Fix $\varepsilon > 0$ arbitrarily, and use the Archimedean property to choose a positive integer n such that $\big(f(b) - f(a)\big)(b - a)/n < \varepsilon$. Let P_n be the equal-length partition with n subintervals. On the ith subinterval $I_i = [t_{i-1}, t_i]$, we have $m_i = f(t_{i-1})$, $M_i = f(t_i)$, and $\Delta t_i = (b - a)/n$ independently of i. Writing Δt instead of Δt_i,

$$U(f, P_n) - L(f, P_n) = \sum_{i=1}^{n} (M_i - m_i)\,\Delta t_i = \left[\sum_{i=1}^{n} \big(f(t_i) - f(t_{i-1})\big)\right] \Delta t.$$

The sum in square brackets telescopes, so

$$U(f, P_n) - L(f, P_n) = \big(f(t_n) - f(t_0)\big)\Delta t = \big(f(b) - f(a)\big)\frac{b - a}{n} < \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, f is integrable on $[a, b]$. □

Integrability of Continuous Functions

In this section, we prove that a continuous function $f : [a, b] \to \mathbf{R}$ is integrable on $[a, b]$. The strategy is to show the upper and lower sums can be made arbitrarily close with a suitable choice of partition, so that f is integrable by Proposition 7.11. We first establish a technical property:

Lemma 7.25. *If $f : [a, b] \to \mathbf{R}$ is a continuous function, then f satisfies the following condition:*

> *For every $\varepsilon > 0$, there exists a $\delta > 0$ such that if t and $t' \in [a, b]$ and $|t - t'| \leq \delta$, then $|f(t) - f(t')| < \varepsilon$.*

Remark 7.26. Careful comparison of this condition with the ε-δ criterion for continuity (Corollary 6.16) reveals an important subtlety: If the domain of the continuous function f is a closed, bounded interval $[a, b]$, then for every $\varepsilon > 0$, there is a *single* δ that simultaneously "wins" the ε-δ game for Player δ at each point of $[a, b]$. A function satisfying this condition is said to be *uniformly continuous on* $[a, b]$.

Proof. We proceed contrapositively. If the stated condition fails, then Player ε can force a win; that is, there exists an $\varepsilon > 0$ such that for every $\delta > 0$, there are points t and t' in $[a, b]$ such that $|t - t'| \leq \delta$ but $|f(t) - f(t')| \geq \varepsilon$. We will prove that in this situation, f is discontinuous at some point of $[a, b]$.

Fix $\varepsilon > 0$ as in the preceding paragraph. For each positive integer n, choose points t_n and t'_n in $[a, b]$ such that $|t_n - t'_n| \leq 1/n$, but $|f(t_n) - f(t'_n)| \geq \varepsilon$. Since $[a, b]$ is a closed, bounded interval, the Bolzano-Weierstrass theorem implies there exists a convergent subsequence $(t_{\nu(k)}) \to t_\infty \in [a, b]$. Since

$$|t'_{\nu(k)} - t_\infty| \leq |t'_{\nu(k)} - t_{\nu(k)}| + |t_{\nu(k)} - t_\infty| \leq 1/k + |t_{\nu(k)} - t_\infty|,$$

the sequence $(t'_{\nu(k)})$ also converges to t_∞. But by construction, the images sequences $\big(f(t_{\nu(k)})\big)$ and $\big(f(t'_{\nu(k)})\big)$ do not have the same limit (even if both limits exist), because $|f(t_{\nu(k)}) - f(t'_{\nu(k)})| \geq \varepsilon$ for all k. By definition, f is discontinuous at t_∞. □

Theorem 7.27. *Let $f : [a, b] \to \mathbf{R}$ be a continuous function. Then f is integrable on $[a, b]$.*

Proof. Fix $\varepsilon > 0$. By Lemma 7.25, there exists a $\delta > 0$ such that if t and t' are elements of $[a, b]$ such that $|t - t'| \leq \delta$, then

$$|f(t) - f(t')| < \varepsilon/(b - a).$$

Use the Archimedean property to pick a positive integer n with $1/n < \delta$, and let $P = P_n$ be the equal-length partition of $[a, b]$ into n subintervals.

For each $i = 1, \dots, n$, if t and $t' \in I_i$, then $|t - t'| \leq 1/n < \delta$, so $|f(t) - f(t')| < \varepsilon/(b - a)$. Since f is continuous, f achieves a maximum value and a minimum value in I_i by the Extreme Value Theorem; thus

$$M_i^f - m_i^f < \varepsilon/(b - a).$$

Summing over i,

$$U(f, P) - L(f, P) = \sum_{i=1}^{n} (M_i^f - m_i^f)\,\Delta t_i < \frac{\varepsilon}{b - a} \sum_{i=1}^{n} \Delta t_i = \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, f is integrable by Proposition 7.11. □

Theorem 7.28. *If f and g are integrable functions on $[a, b]$, then the product fg is integrable on $[a, b]$.*

Remark 7.29. There is no general formula for the integral of fg in terms of the integrals of f and g.

Proof. By the "polarization identity"

$$fg = \tfrac{1}{2}\big((f+g)^2 - f^2 - g^2\big),$$

it suffices to prove that if f is integrable, then f^2 is integrable.

Fix $\varepsilon > 0$. Since f is integrable, there exists a positive real number C such that $|f(t)| \leq C$ for all t in $[a, b]$, and there exists a partition P of $[a, b]$ such that

$$U(f, P) - L(f, P) < \varepsilon/(2C).$$

Suppose t and t' are elements of some subinterval I_i. By the difference of squares identity,

$$|f(t)^2 - f(t')^2| = |f(t) + f(t')| \cdot |f(t) - f(t')| \leq 2C \cdot |f(t) - f(t')|.$$

Taking the supremum over t, then the infimum over t',

$$M_i^{f^2} - m_i^{f^2} \leq 2C \cdot (M_i^f - m_i^f) \quad \text{for } i = 1, \dots, n.$$

Summing over i,

$$U(f^2, P) - L(f^2, P) \leq 2C \cdot \big(U(f, P) - L(f, P)\big) < \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, f^2 is integrable on $[a, b]$. □

7.4 Definite Integrals

Definition 7.30. Let $f : [a, b] \to \mathbf{R}$ be integrable. For each x in $[a, b]$, the function f is integrable on $[a, x]$. The function $F : [a, b] \to \mathbf{R}$ defined by

$$F(x) = \int_a^x f = \int_a^x f(t)\, dt$$

is called the *definite integral* of f from a.

Example 7.31. Let $\operatorname{sgn} : \mathbf{R} \to \mathbf{R}$ be the signum function, defined by $\operatorname{sgn}(t) = t/|t|$ if $t \neq 0$, and $\operatorname{sgn}(0) = 0$. We claim the signum function is integrable on every closed, bounded interval, and

$$\int_0^x \operatorname{sgn}(t)\, dt = |x| \quad \text{for all real } x.$$

To prove sgn is integrable, note that if $x > 0$, then on the interval $[0, x]$, we have $\operatorname{sgn}(t) = 1$ except at one point. Since a constant function is integrable and sgn differs from a constant function at only one point, sgn is integrable on $[0, x]$ by Corollary 7.18, and

$$\int_0^x \operatorname{sgn}(t)\,dt = \int_0^x 1\,dt = x - 0 = x \quad \text{for all } x \geq 0.$$

An entirely similar argument handles the interval $[x, 0]$ if $x < 0$, and we have

$$\int_0^x \operatorname{sgn}(t)\,dt = \int_0^x (-1)\,dt = -1(x - 0) = -x = |x| \quad \text{for all } x < 0.$$

The patching property guarantees sgn is integrable on an arbitrary closed, bounded interval.

Example 7.32. If f is integrable on some interval containing a, x, and y, and F is the definite integral of f, the cocycle property gives

$$F(y) - F(x) = \int_a^y f - \int_a^x f = \int_x^y f.$$

Theorem 7.33. *Let f be integrable on $[a, b]$, and let*

$$F(x) = \int_a^x f$$

be the definite integral of f from a.

(i) *If $|f(t)| \leq M$ for all but finitely many t, then*

$$\bigl|F(y) - F(x)\bigr| \leq M|y - x| \quad \text{for all } x,\ y \text{ in } [a, b].$$

In particular, F is continuous.

(ii) *If f is non-negative, then F is non-decreasing.*

(iii) *If f is continuous and positive except perhaps at finitely many points, then F is strictly increasing on $[a, b]$.*

Proof. (i) We may as well assume $|f(t)| \leq M$ for all t, since by Corollary 7.18 we may change the value of f at finitely many points without changing the integral. If $a \leq x \leq y \leq b$, then

$$\bigl|F(y) - F(x)\bigr| = \left|\int_x^y f\right| \leq \int_x^y |f| \leq \int_x^y M = M|y - x|.$$

To prove F is continuous, fix $\varepsilon > 0$ arbitrarily, and let $\delta = \varepsilon/(M+1)$. (Adding 1 accommodates the possibility that $M = 0$.) If $|y-x| < \delta$, then $|F(y) - F(x)| \leq M|y-x| < M\delta < \varepsilon$.

(ii) Suppose f is non-negative. If $a \leq x < y \leq b$, then

$$0 = \int_x^y 0\,dt \leq \int_x^y f(t)\,dt = F(y) - F(x), \quad \text{i.e., } F(x) \leq F(y).$$

(iii) Let $a \leq x < y \leq b$. Since f is continuous and positive except perhaps at finitely many points, there exists a point t_0 with $x < t_0 < y$ such that $0 < f(t_0)$. By Corollary 6.17 (ii), there exists a $\delta > 0$ such that if $|t - t_0| < \delta$, then $f(t) > f(t_0)/2$. Shrinking δ if necessary, we may assume $[t_0 - \delta, t_0 + \delta] \subseteq (x, y)$. By monotonicity,

$$0 < (2\delta)\bigl(f(t_0)/2\bigr) = \int_{t_0-\delta}^{t_0+\delta} f(t_0)/2\,dt \leq \int_{t_0-\delta}^{t_0+\delta} f(t)\,dt \leq \int_x^y f(t)\,dt,$$

or $0 < F(y) - F(x)$. That is, F is strictly increasing. ☐

Remark 7.34. The preceding proof only uses the fact that there exists no interval on which f is identically equal to 0.

Corollary 7.35. *If $k \geq 0$ is an integer and x, y are real numbers, then*

$$\int_x^y t^k\,dt = \frac{y^{k+1} - x^{k+1}}{k+1}.$$

Proof. If $x < y$ are positive, this restates Example 7.15.

Since each side is continuous in x, taking limits as $x \to 0$ gives

$$\int_0^y t^k\,dt = \frac{y^{k+1}}{k+1} \quad \text{for all } y \geq 0.$$

If $w \leq 0$, so $0 \leq -w$, then taking $\mu = -1$ in Theorem 7.22 gives

$$\int_0^w t^k\,dt = -\int_0^{-w} (-t)^k\,dt = (-1)^{k+1} \int_0^{-w} t^k\,dt = \frac{w^{k+1}}{k+1}.$$

We have established that $F(w)$, the integral of t^k from 0 to w, is equal to $w^{k+1}/(k+1)$ for all real w. If x, y are arbitrary, then

$$\int_x^y t^k\,dt = F(y) - F(x) = \frac{y^{k+1} - x^{k+1}}{k+1}.$$ ☐

Example 7.36. Let

$$p(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_n t^n = \sum_{k=0}^{n} c_k t^k$$

be a polynomial. Since p is continuous on $\mathbf{R}$, p is integrable on every interval $[0, x]$, and

$$\int_0^x p(t)\,dt = \int_0^x \sum_{k=0}^{n} c_k t^k\,dt = \sum_{k=0}^{n} c_k \int_0^x t^k\,dt = \sum_{k=0}^{n} c_k \frac{x^{k+1}}{k+1}.$$

If we call this new polynomial $P(x)$, then

$$\int_a^b p(t)\,dt = \int_0^b p(t)\,dt - \int_0^a p(t)\,dt = P(b) - P(a).$$

Remark 7.37. To evaluate a definite integral from the definition for even a single x, we must compute the supremum of the set of lower sums of f as P ranges over the set of partitions of $[a, x]$. In general this is somewhere between laborious and genuinely difficult.

The preceding example shows that for a polynomial function, the definite integral (laborious to evaluate) is equal to a particular polynomial (trivial to evaluate). That these functions are one and the same is a substantial and non-trivial piece of information.

The Natural Logarithm

One of the most important functions in analysis may be defined as a definite integral:

Definition 7.38. The *natural logarithm* function $\log : (0, \infty) \to \mathbf{R}$ is defined for $x > 0$ by

$$\log x = \int_1^x \frac{dt}{t}.$$

Theorem 7.39. *For all positive real numbers x and y,*

$$\log xy = \log x + \log y, \qquad \log(1/x) = -\log x.$$

The natural logarithm is strictly increasing and surjective, and there exists a unique real number e, with $2 < e < 3$, such that $\log e = 1$.

Proof. If x and y are arbitrary positive real numbers, then

$$\begin{aligned}\log xy = \int_1^{xy} \frac{dt}{t} &= \int_1^x \frac{dt}{t} + \int_x^{xy} \frac{dt}{t} && \text{Patching}\\ &= \int_1^x \frac{dt}{t} + x\int_1^y \frac{dt}{xt} && \text{Theorem 7.22 with } \mu = x\\ &= \log x + \log y.\end{aligned}$$

Taking $y = 1/x$, we have $\log x + \log(1/x) = \log 1 = 0$, which proves $\log(1/x) = -\log x$.

The logarithm is strictly increasing because the integrand $f(t) = 1/t$ is continuous and positive. In particular, there is at most one positive real x such that $\log x = 1$.

To prove there exists a real number e, with $2 < e < 3$, such that $\log e = 1$, it suffices by the Intermediate Value Theorem to show

$$\log 2 = \int_1^2 \frac{1}{t}\,dt < 1 < \int_1^3 \frac{1}{t}\,dt = \log 3.$$

The first inequality is straightforward: $1/t \leq 1$ on the interval $[1, 2]$, with equality only at the left endpoint, so

$$\log 2 = \int_1^2 \frac{1}{t}\,dt < \int_1^2 1\,dt = 2 - 1 = 1.$$

The second inequality is a bit sneakier: Since

$$0 \leq \frac{(t-2)^2}{4t} = \frac{t^2 - 4t + 4}{4t} = (t/4) - 1 + 1/t$$

for all $t > 0$, and equality holds only at $t = 2$,

$$0 < \int_1^3 \big((t/4) - 1 + (1/t)\big)\,dt = \frac{3^2 - 1^2}{4 \cdot 2} - (3 - 1) + \log 3 = -1 + \log 3,$$

or $1 < \log 3$.

A straightforward inductive argument shows that for every non-negative integer n, $\log e^n = n$, and since $\log(1/x) = -\log x$, we have $\log e^{-n} = -\log e^n = -n$ as well.

If k is an arbitrary real number, the Archimedean property guarantees that there exists a positive integer n with $|k| < n$. Since

$$\log(e^{-n}) = -n < -|k| \leq k \leq |k| < n = \log(e^n),$$

the Intermediate Value Theorem applied to log on the interval $[e^{-n}, e^n]$ shows there exists a real x with $\log x = k$; that is, log is surjective. □

Integration and O Notation

Theorem 7.40. *Assume f is integrable on some interval containing x_0, and let*

$$F(x) = \int_{x_0}^{x} f(t)\,dt$$

be the definite integral of f from x_0. If there is a positive integer k such that $f(t) \approx O(t-x_0)^k$ for $t \approx x_0$, then $F(x) \approx O(x-x_0)^{k+1}$ for $x \approx x_0$.

Proof. By hypothesis, there exist positive real numbers C and δ such that $|f(t)| \leq C|t-x_0|^k$ for $|t-x_0| < \delta$. For $x_0 \leq x < x_0 + \delta$,

$$|F(x)| \leq \int_{x_0}^{x} |f(t)|\,dt \leq C\int_{x_0}^{x} (t-x_0)^k\,dt = \frac{C}{k+1}(x-x_0)^{k+1}.$$

If instead $x_0 - \delta < x \leq x_0$,

$$|F(x)| \leq \int_{x}^{x_0} |f(t)|\,dt \leq C\int_{x}^{x_0} (x_0-t)^k\,dt = \frac{C}{k+1}|x-x_0|^{k+1}.$$

Thus $F(x) \approx O(x-x_0)^{k+1}$ for $x \approx x_0$. □

7.5 Integration of Power Series

Theorem 7.41. *Suppose we have a power series with radius $R > 0$:*

$$\begin{aligned} f(x) &= \sum_{k=0}^{\infty} a_k(x-x_0)^k \\ &= a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \dots . \end{aligned}$$

For every x with $|x-x_0| < R$, f is integrable between x_0 and x, and

$$\begin{aligned} \int_{x_0}^{x} f(t)\,dt &= \sum_{k=0}^{\infty} \frac{a_k}{k+1}\,(x-x_0)^{k+1} \\ &= a_0(x-x_0) + \frac{a_1}{2}(x-x_0)^2 + \frac{a_2}{3}(x-x_0)^3 + \dots . \end{aligned}$$

Proof. As usual, assume $x_0 = 0$. Theorem 6.40 implies f is continuous on $(-R, R)$, and therefore integrable on every closed, bounded subinterval by Theorem 7.27.

For each $n \geq 0$, introduce the polynomials

$$f_n(t) = \sum_{k=0}^{n} a_k t^k, \qquad F_n(x) = \int_0^x f_n(t)\,dt = \sum_{k=0}^{n} \frac{a_k}{k+1}\,x^{k+1},$$

and define

$$F(x) = \int_0^x f(t)\,dt, \qquad G(x) = \sum_{k=0}^{\infty} \frac{a_k}{k+1}\,x^{k+1}.$$

The power series $G(x)$ converges absolutely for every x with $|x| < R$, by comparison with the series $xf(x) = \sum_k a_k x^{k+1}$.

To prove that $F(x) = G(x)$ for all x with $|x| < R$, it suffices to show that if $0 < r < R$ and $\varepsilon > 0$ are arbitrary, then $F - G \approx \mathsf{A}(\varepsilon)$ on $[-r, r]$.

Fix $r < R$ and $\varepsilon > 0$. By Proposition 6.44, there exists an index N such that

$$f \approx f_N + \mathsf{A}\big(\varepsilon/(2r)\big) \quad \text{on } [-r, r].$$

If $x \in [-r, r]$ is arbitrary, then

$$|F(x) - F_N(x)| \leq \left| \int_0^x |f(t) - f_N(t)|\,dt \right| < |x|\,\varepsilon/(2r) < \varepsilon/2.$$

That is, $F - F_N \approx \mathsf{A}(\varepsilon/2)$ on $[-r, r]$. Furthermore,

$$|G(x) - F_N(x)| \leq \sum_{k=N+1}^{\infty} \left| \frac{a_k}{k+1} \right| |x|^{k+1} \leq |r| \sum_{k=N+1}^{\infty} |a_k|\,|r|^k < \varepsilon/2.$$

That is, $F_N - G \approx \mathsf{A}(\varepsilon/2)$ on $[-r, r]$. Adding these estimates,

$$F - G = (F - F_N) + (F_N - G) \approx \mathsf{A}(\varepsilon) \quad \text{on } [-r, r].$$

Since $\varepsilon > 0$ was arbitrary, $F \equiv G$ on $[-r, r]$. Since $r < R$ was arbitrary, $F(x) = G(x)$ for all x with $|x| < R$. □

Example 7.42. Fix $x_0 > 0$. The geometric series

$$\frac{1}{x} = \frac{1}{x_0\big[1 + (x - x_0)/x_0\big]} = \sum_{k=0}^{\infty} \frac{(-1)^k}{x_0^{k+1}} (x - x_0)^k$$

(Example 5.61) converges on the interval $|x - x_0| < x_0$, i.e., on $(0, 2x_0)$. Integrating term by term from x_0 to x,

$$\log x - \log x_0 = \sum_{k=0}^{\infty} \frac{(-1)^k}{x_0^{k+1}}\, \frac{(x - x_0)^{k+1}}{k+1} \quad \text{for } 0 < x < 2x_0.$$

Exercises

Exercise 7.1. *Using only the definition of the integral*, prove that if f is a non-negative integrable function on $[a, b]$, then $\int_a^b f \geq 0$. Use this result to prove Theorem 7.16 (ii).

Exercise 7.2. Show that

$$\int_0^x |t|\,dt = \frac{x|x|}{2} \quad \text{for all real } x.$$

Suggestion: Separate the cases $x \geq 0$ and $x < 0$.

Exercise 7.3. Let f be integrable on $[a, b]$. Prove there exists an x with $a \leq x \leq b$ such that

$$\int_a^x f(t)\,dt = \int_x^b f(t)\,dt.$$

Is it generally possible to choose $a < x < b$?

Exercise 7.4. Let f be a function on $[0, 1]$, and assume for every δ with $0 < \delta < 1$ that f is integrable on $[\delta, 1]$. Give a proof or counterexample to each of the following:

(a) If $\lim\limits_{\delta \to 0} \int_\delta^1 f(x)\,dx$ exists, then f is integrable on $[0, 1]$.

(b) If $\lim(f, 0^+)$ exists, then f is integrable on $[0, 1]$.

Exercise 7.5. Let $a < b$ be real numbers, $P = \{t_i\}_{i=0}^n$ a partition of $[a, b]$, and f a bounded function on $[a, b]$. A *Riemann sum* of f with respect to P is an expression of the form

$$S = \sum_{i=1}^n f(t_i^*)\,\Delta t_i, \quad t_{i-1} \leq t_i^* \leq t_i \text{ for } i = 1, \ldots, n.$$

(a) Show that if S is a Riemann sum of f with respect to P, then $L(f, P) \leq S \leq U(f, P)$.

(b) Assume f is continuous on $[a, b]$. By Lemma 7.25, if $\varepsilon > 0$ is arbitrary, there exists a $\delta > 0$ such that if x and y are points in $[a, b]$ and $|x - y| < \delta$, then $|f(x) - f(y)| < \varepsilon/2(b - a)$. Prove that if P is a partition such that $\Delta t_i < \delta$ for all i, then

$$\left| S - \int_a^b f \right| < \varepsilon \quad \text{for every Riemann sum } S \text{ of } f \text{ with respect to } P.$$

(c) Assume f is continuous. Let $n \geq 1$ be an integer, and let $P = P_n$ be the partition of $[a, b]$ into n subintervals of length $\Delta t = (b-a)/n$. Prove that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} f(a + i\,\Delta t) = \frac{1}{b-a} \int_a^b f(t)\,dt.$$

(The integral is called the *average value* of f on the interval $[a, b]$.) Hint: The sum on the left is a Riemann sum for f with respect to P, the *right-hand sum.*

Exercise 7.6. Let $a < b$ be real numbers.

(a) Calculate the average value of $f(x) = x$ over $[a, b]$.

(b) Show the average value of $f(x) = x^2$ over $[a, b]$ is $(a^2 + ab + b^2)/3$.

(c) Let $n > 0$ be an integer. Calculate the average value of $f(x) = x^n$ over $[a, b]$. Express your answer in a form that reduces, when $n = 2$, to the formula in part (b).

Exercise 7.7. Suppose f is integrable on $[a, b]$, and let

$$\overline{f} = \frac{1}{b-a} \int_a^b f(t)\,dt$$

be the average value of f on $[a, b]$.

(a) Prove that the integral of $f - \overline{f}$ over $[a, b]$ is equal to 0.

(b) (The *Mean Value Theorem for integrals*) Prove that if f is continuous on $[a, b]$, there exists a number c in (a, b) such that $f(c) = \overline{f}$.

(c) If $f(x) = x^2$ on $[0, 1]$, find the value of c in $(0, 1)$ that satisfies the Mean Value Theorem for integrals, and carefully sketch f and its average value.

Exercise 7.8. Let f be integrable on some open ball $B_r(x_0)$.

(a) Prove that if $0 < |h| < r$, then

$$\frac{1}{h} \int_{x_0}^{x_0+h} f$$

is the average value of f on the closed interval with endpoints x_0 and $x_0 + h$ (even if $h < 0$).

(b) Assume f is continuous at x_0. Prove that

$$\lim_{h \to 0} \frac{1}{h} \int_{x_0}^{x_0+h} f(t)\, dt = f(x_0).$$

(c) Show by example that the result of part (b) is generally false if f is discontinuous at x_0.

Exercise 7.9. Suppose $|f|$ is integrable on $[a, b]$. Is f necessarily integrable? Give a proof or counterexample.

Exercise 7.10. Let $P = \{t_i\}_{i=1}^{n}$ be a partition of $[a, b]$. The points t_{i-1}, $\bar{t}_i = \frac{1}{2}(t_{i-1} + t_i)$, and t_i are called, respectively, the *left-hand endpoint*, the *midpoint*, and the *right-hand endpoint*. The corresponding Riemann sums (see Exercise 7.5) are the *left* sum $\mathrm{LEFT}(f, P)$, the *midpoint* sum $\mathrm{MID}(f, P)$, and the *right* sum $\mathrm{RIGHT}(f, P)$. The average of the left and right sums is the *trapezoid* sum $\mathrm{TRAP}(f, P)$.

(a) Prove that if f is non-decreasing on $[a, b]$, then for every partition P,

$$\mathrm{LEFT}(f, P) \le \int_a^b f \le \mathrm{RIGHT}(f, P).$$

What inequalities hold if f is non-increasing?

(b) Show that $\mathrm{TRAP}(f, P)$ is the sum of the areas of the trapezoids bounded by the secant lines joining the points $\big(t_i, f(t_i)\big)$.

(c) Prove that $\mathrm{MID}(f, P)$ is the sum of the areas of the trapezoids bounded by arbitrary non-vertical lines passing through the points $\big(\bar{t}_i, f(\bar{t}_i)\big)$.
Suggestion: Show that every such trapezoid has area $f(\bar{t}_i)\, \Delta t_i$.

Exercise 7.11. In each part, let $f(x) = x^2$ on $[a, b]$, and let $P = \{a, b\}$ be the partition with one subinterval.

(a) Calculate the left, right, trapezoid, and midpoint sums, and illustrate each sum with a sketch.

(b) Show that $\big(\mathrm{TRAP}(f, P) + 2\,\mathrm{MID}(f, P)\big)/3$ is the exact value of the integral.

Exercise 7.12. In each part, let $f(x) = x^3$ on $[a, b]$, and let $P = \{a, b\}$ be the partition with one subinterval.

(a) Calculate the left, right, trapezoid, and midpoint sums.

(b) By how much does $\bigl(\mathrm{TRAP}(f,P) + 2\,\mathrm{MID}(f,P)\bigr)/3$ differ from the exact value of the integral?

Exercise 7.13. Let ρ be a continuous, *non-negative* function on an interval $[a,b]$. The *weighted average* of an integrable function f is defined to be

$$\left(\int_a^b f\rho\right) \Bigg/ \left(\int_a^b \rho\right).$$

State and prove the analog of the Mean Value Theorem for integrals for weighted averages.

Exercise 7.14. Let $0 < a < b$ be real, and let $\rho(x) = x$. Calculate the weighted average of each of the following on $[a,b]$:
(a) $f(x) = 1$. (b) $f(x) = x$. (c) $f(x) = x^2$. (d) $f(x) = x^3$.

Chapter 8

Differentiation

8.1 Differentiability

Definition 8.1. Let X be a non-empty set of real numbers, $f : X \to \mathbf{R}$ a function, and $x_0 \in X$ an interior point; that is, there exists a $\delta > 0$ such that $B_\delta(x_0) \subseteq X$.

For $0 < |h| < \delta$, put $x = x_0 + h$, and define the *difference quotient*

$$\Delta f(x_0, h) = \frac{f(x_0 + h) - f(x_0)}{h} = \frac{f(x) - f(x_0)}{x - x_0}.$$

If $\Delta f(x_0, h)$ has a limit as $h \to 0$, we say f is *differentiable at* x_0, and call the limit

$$f'(x_0) = \lim_{h \to 0} \Delta f(x_0, h) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

the *derivative* of f at x_0.

If X is open and f is differentiable at x_0 for each x_0 in X, we say f is *differentiable on* X.

Remark 8.2. The difference quotient $\Delta f(x_0, h)$ represents the "average rate of change" of f on the interval between x_0 and $x = x_0 + h$, namely, the change in f on this interval divided by the length of the interval.

To say f is differentiable at x_0 means these rates of change approach a finite limit as $h \to 0$, in which case the derivative represents the "instantaneous rate of change" of f at x_0.

Example 8.3. If f is a constant function on some open interval X, then f is differentiable on X, and $f'(x_0) = 0$ for all x_0. Indeed, every difference quotient of f is zero.

Example 8.4. The signum function sgn is differentiable on the set $X = \mathbf{R} \setminus \{0\}$ of non-zero real numbers, and $\operatorname{sgn}'(x) = 0$ for all x in X, but sgn is not constant.

Example 8.5. The identity function ι is differentiable on every open interval, and since every difference quotient of ι is equal to 1, we have $\iota'(x_0) = 1$ for all x_0.

Example 8.6. A quadratic polynomial $f(x) = a_0 + a_1x + a_2x^2$ is differentiable on $\mathbf{R}$, and $f'(x_0) = a_1 + 2a_2x_0$. To prove this, we calculate

$$\begin{aligned}\frac{f(x) - f(x_0)}{x - x_0} &= \frac{(a_0 + a_1x + a_2x^2) - (a_0 + a_1x_0 + a_2x_0^2)}{x - x_0} \\ &= \frac{a_1(x - x_0) + a_2(x^2 - x_0^2)}{x - x_0} = a_1 + a_2(x + x_0),\end{aligned}$$

provided $x \neq x_0$. Fixing x_0 and taking the limit as $x \to x_0$, we have

$$f'(x_0) = \lim_{x\to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x\to x_0} a_1 + a_2(x + x_0) = a_1 + 2a_2x_0.$$

Example 8.7. The square root function $f(x) = \sqrt{x}$ is differentiable on $(0, \infty)$, and $f'(x_0) = 1/(2\sqrt{x_0})$. Fix $x_0 > 0$; for $|h| < x_0$, we have

$$\begin{aligned}\Delta f(x_0, h) &= \frac{\sqrt{x_0 + h} - \sqrt{x_0}}{h} = \frac{\sqrt{x_0 + h} - \sqrt{x_0}}{h} \cdot \frac{\sqrt{x_0 + h} + \sqrt{x_0}}{\sqrt{x_0 + h} + \sqrt{x_0}} \\ &= \frac{(x_0 + h) - x_0}{h(\sqrt{x_0 + h} + \sqrt{x_0})} = \frac{1}{\sqrt{x_0 + h} + \sqrt{x_0}}.\end{aligned}$$

Since the square root function is continuous,

$$f'(x_0) = \lim_{h\to 0} \Delta f(x_0, h) = \lim_{h\to 0} \frac{1}{\sqrt{x_0 + h} + \sqrt{x_0}} = \frac{1}{2\sqrt{x_0}}.$$

The square root function is not differentiable at 0. First, 0 is not an interior point of the domain $X = [0, \infty)$. However, even if we let $h \to 0$ from the right, the difference quotient $\Delta f(0, h) = \sqrt{h}/h = 1/\sqrt{h}$ diverges to ∞.

Local Extrema

Definition 8.8. Let $f : X \to \mathbf{R}$, and let $x_0 \in X$ be a limit point. We say x_0 is a *local minimum* of f in X if there exists a $\delta > 0$ such that

$f(x_0) \leq f(x)$ whenever $x \in X$ and $|x - x_0| < \delta$. The real number $f(x_0)$ is called a *local minimum value* of f in X.

If instead $|x - x_0| < \delta$ implies $f(x_0) < f(x)$, we say x_0 is a *strict local minimum* of f.

Analogously, we define *local maximum* and *strict local maximum* points of X. A *local extremum* of f in X is a point that is either a local minimum or local maximum.

Example 8.9. On the set $X = [0, 1]$, the function $f(x) = x$ has a local minimum at 0 and a local maximum at 1.

On the open interval $(0, 1)$, the function $f(x) = x$ is bounded, but has no local extrema.

Proposition 8.10. *If* $f : [a, b] \to \mathbf{R}$ *is a continuous function, and* x_0 *is a local extremum of* f *in* X*, then* x_0 *is one of the following:*

(i) *An endpoint of* $[a, b]$*.*

(ii) *An interior point at which* $f'(x_0)$ *does not exist.*

(iii) *An interior point at which* $f'(x_0) = 0$*.*

Proof. Suppose x_0 is a local extremum. If x_0 is an endpoint, or if $f'(x_0)$ does not exist, there is nothing to prove. Assume, therefore, that $x_0 \in (a, b)$, and

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

exists. We may assume without loss of generality that x_0 is a local minimum; if not, replace f by $-f$. By definition of a local minimum, $f(x_0) \leq f(x)$ for $x \approx x_0$. That is, the numerator of the difference quotient is non-negative.

Letting $x \to x_0$ from the left, i.e., taking $x - x_0 < 0$, we find that $f'(x_0) \leq 0$. Letting $x \to x_0$ from the right, i.e., taking $x - x_0 > 0$, we find that $f'(x_0) \geq 0$. Thus $f'(x_0) = 0$. □

Derivatives and *o*-Notation

Proposition 8.11. *A function* f *is differentiable at an interior point* x_0 *if and only if there exists a real number* m *such that*

$$f(x_0 + h) \approx f(x_0) + mh + o(h) \quad \text{for } h \approx 0,$$

and in this event, $m = f'(x_0)$*.*

Proof. By definition, f is differentiable at x_0 and $m = f'(x_0)$ if and only if

$$0 = \lim_{h\to 0} \frac{f(x_0+h) - f(x_0)}{h} - m = \lim_{h\to 0} \frac{f(x_0+h) - f(x_0) - mh}{h}.$$

Again by definition, $f(x_0+h) - f(x_0) - mh \approx o(h)$ for $h \approx 0$. ☐

Remark 8.12. If f is differentiable at x_0, the first-degree polynomial $p_1(x) = f(x_0) + f'(x_0)(x - x_0)$ is called the *linear approximation* of f at x_0.

The linear approximation is the *unique* polynomial p of degree 1 satisfying $f(x) \approx p(x) + o(x - x_0)$ for $x \approx x_0$.

Proposition 8.13. *If f is differentiable at x_0, then f is continuous at x_0.*

Proof. Let f be differentiable at x_0. By Proposition 8.11,

$$\begin{aligned} f(x) &\approx f(x_0) + f'(x_0)(x - x_0) + o(x - x_0) \\ &\approx f(x_0) + o(1) \quad \text{for } x \approx x_0. \end{aligned}$$

☐

Example 8.14. The converse of Proposition 8.13 is false. The absolute value function, $f(x) = |x|$, is continuous everywhere, but not differentiable at 0:

$$\Delta f(0, h) = \frac{f(h) - f(0)}{h - 0} = \frac{|h|}{h} = \operatorname{sgn} h,$$

which has no limit at 0. As noted in Chapter 6, a "typical" continuous function is *nowhere* differentiable.

Geometric Interpretation

Definition 8.15. Let f be a real-valued function defined on some ball I centered at x_0.

If $x_1 \in I$ and $x_1 \neq x_0$, the line with equation

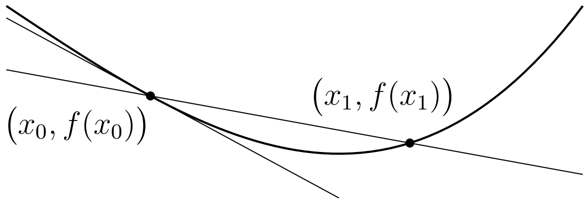
$$y = f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0)$$

is called the *secant line* through $\big(x_0, f(x_0)\big)$ and $\big(x_1, f(x_1)\big)$.

If f is differentiable at x_0, the line with equation

$$y = f(x_0) + f'(x_0)(x - x_0)$$

is called the *tangent line* through $\big(x_0, f(x_0)\big)$.

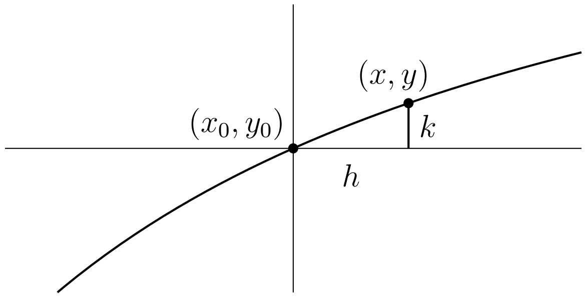


Geometrically, the secant line through $\big(x_0, f(x_0)\big)$ and $\big(x_1, f(x_1)\big)$ "approaches" the tangent line through $\big(x_0, f(x_0)\big)$ as $x_1 \to x_0$.

Proposition 8.11 also has a geometric interpretation: Differentiating a function f at x_0 amounts to "zooming in on the graph with factor infinity." To justify this claim, write $y_0 = f(x_0)$ and introduce Cartesian coordinates (h, k) with origin (x_0, y_0). That is, put $x = x_0 + h$ and $y = y_0 + k$. In the (h, k) coordinate system, the graph of f has equation $k = f(x_0 + h) - f(x_0)$. Zooming in by a factor of λ sends (h, k) to $(\lambda h, \lambda k) = (\tilde{h}, \tilde{k})$. The equation of the graph in the scaled coordinates $(\tilde{h}, \tilde{k})$ is therefore $\tilde{k}/\lambda = f(x_0 + \tilde{h}/\lambda) - f(x_0)$, or (using (h, k) to denote the *scaled* coordinates)

$$k = \left(\frac{f(x_0 + h/\lambda) - f(x_0)}{(h/\lambda)}\right) \cdot h.$$

Now, $h/\lambda \to 0$ as $\lambda \to \infty$, so if f is differentiable at x_0, the equation of the graph in scaled coordinates "approaches" $k = f'(x_0) \cdot h$, the equation of the tangent line in the original (h, k) coordinate system.



Loosely, the graph of a differentiable function is made up of infinitely many infinitesimal line segments, and the slope of the segment at x_0 is $f'(x_0)$, the "rise over the run at x_0."

Remark 8.16. A functional relation $y = f(x)$ determines the "dependent variable" y in terms of the "independent variable" x. In *Leibniz notation*, a difference quotient is denoted $\Delta y/\Delta x$. If f is differentiable, the limit of $\Delta y/\Delta x$ as $\Delta x \to 0$ is denoted dy/dx rather than $f'(x)$, and is viewed as a (formal) ratio of infinitesimals. Computational formulas below may be viewed as granting license to manipulate infinitesimal ratios as if they were ordinary (real) fractions, see Exercise 8.11.

Derivatives as Functions

So far, we have concentrated on differentiability at a single point. Ultimately, we are interested in functions that are differentiable everywhere in their domain. From now on, our notation changes to reflect this.

Definition 8.17. Let X be an open set in $\mathbf{R}$. A function $f : X \to \mathbf{R}$ is *differentiable on* X if $f'(x)$ exists for every x in X. The function $f' : X \to \mathbf{R}$ is called the (*first*) *derivative* of f, and f is said to be an *antiderivative* of f'. If f' is a continuous function, we say f is *continuously differentiable on* X.

If the derivative f' is itself differentiable, we say f is *twice differentiable on* X, and denote the derivative of f' by f''. The function $f'' : X \to \mathbf{R}$ is called the *second derivative* of f. If the second derivative of f is continuous, we say f is *twice continuously differentiable on* X.

Higher-order derivatives are defined inductively. We say f is n *times differentiable on* X if the nth derivative function $f^{(n)}$ exists at each point of X, and we say f is n *times continuously differentiable* if $f^{(n)}$ is continuous on X.

If f is n times differentiable for every $n \geq 1$, we say f is *infinitely differentiable*, or *smooth*, on X.

Remark 8.18. Not every differentiable function is continuously differentiable, see Exercise 8.15. For simplicity, most functions we work with are smooth. The nth derivative of a smooth function is automatically continuous by Proposition 8.13.

In more advanced work, including applications to differential equations, the degree of differentiability is an important detail. A function f is said to be *of class* $\mathscr{C}^n$ on X if f is n times continuously differentiable on X. A smooth function is said to be *of class* $\mathscr{C}^\infty$ on X.

8.2 Differentiation Rules

Theorem 8.19 (Linearity of the derivative). *Suppose f and g are differentiable at x, and let c be a real number. The functions $f \pm g$ and cf*

are differentiable at x, and

$$(f \pm g)'(x) = f'(x) \pm g'(x), \qquad (cf)'(x) = c\,f'(x).$$

Proof. By hypothesis, there exist real numbers $f'(x)$ and $g'(x)$ such that

$$\left.\begin{aligned} f(x+h) &\approx f(x) + h\,f'(x) + o(h) \\ g(x+h) &\approx g(x) + h\,g'(x) + o(h) \end{aligned}\right\} \quad \text{for } h \approx 0.$$

Adding,

$$(f+g)(x+h) \approx (f+g)(x) + h\big(f'(x) + g'(x)\big) + o(h).$$

By Proposition 8.11, this simultaneously proves $f+g$ is differentiable at x and shows the derivative is $f'(x) + g'(x)$.

The proofs for $f - g$ and cf are entirely similar. □

Theorem 8.20 (The product and quotient rules)**.** *If f and g are differentiable at x, then fg is differentiable at x, and*

$$(fg)'(x) = f'(x)g(x) + f(x)g'(x).$$

If $g(x) \neq 0$, then f/g is differentiable at x, and

$$\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}.$$

Proof. By hypothesis,

$$\left.\begin{aligned} f(x+h) &\approx f(x) + h\,f'(x) + o(h) \\ g(x+h) &\approx g(x) + h\,g'(x) + o(h) \end{aligned}\right\} \quad \text{for } h \approx 0.$$

Multiplying these approximations gives

$$\begin{aligned} (fg)(x+h) &= \big[f(x) + h\,f'(x) + o(h)\big]\big[g(x) + h\,g'(x) + o(h)\big] \\ &= (fg)(x) + h\big[f'(x)g(x) + f(x)g'(x)\big] + o(h). \end{aligned}$$

By Proposition 8.11, this establishes the product rule.

For quotients, algebra and the preceding substitutions give

$$\begin{aligned} \Delta(f/g)(x,h) &= \frac{1}{h}\left[\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}\right] \\ &= \frac{1}{h}\left[\frac{f(x+h)g(x) - f(x)g(x+h)}{g(x+h)g(x)}\right] \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{h}\left[\frac{h\big(f'(x)g(x) - f(x)g'(x)\big) + o(h)}{g(x+h)g(x)}\right] \\
&= \frac{f'(x)g(x) - f(x)g'(x)}{g(x+h)g(x)} + o(1).
\end{aligned}$$

Since g is differentiable at x, g is continuous at x, so $g(x+h) \to g(x)$ as $h \to 0$. ☐

Corollary 8.21. *Let $n \geq 0$ be an integer.*

(i) *$f(x) = x^n$ is differentiable for all real x, and $f'(x) = nx^{n-1}$.*

(ii) *$g(x) = x^{-n}$ is differentiable for all $x \neq 0$, and $g'(x) = -nx^{-n-1}$.*

Proof. (i) Here we let $x^0 = 1$ and $0x^{-1} = 0$ even if $x = 0$. With this understanding, $f(x) = x^0$ is a constant function, so $f'(x) = 0 = nx^{n-1}$ for all x if $n = 0$.

When $n = 1$, $f(x) = x$ is the identity function ι. As noted above, $f'(x) = 1 = nx^{n-1}$ for all real x.

Assume inductively for some $n \geq 1$ that the derivative of $f(x) = x^n$ is $f'(x) = nx^{n-1}$. Taking $g(x) = x$ and using the product rule, the derivative of $(fg)(x) = x^{n+1}$ is

$$(fg)'(x) = f'(x)g(x) + f(x)g'(x) = (nx^{n-1}) \cdot x + x^n \cdot 1 = (n+1)x^n.$$

(ii) Let $g(x) = x^{-n} = 1/x^n = 1/f(x)$. By the quotient rule,

$$g'(x) = \left(\frac{1}{f}\right)'(x) = -\frac{f'(x)}{f(x)^2} = -\frac{nx^{n-1}}{(x^n)^2} = -nx^{-n-1}.$$ ☐

Corollary 8.22. *A polynomial function*

$$p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n = \sum_{k=0}^{n} a_kx^k$$

is differentiable on $\mathbf{R}$, and

$$\begin{aligned}
p'(x) &= a_1 + 2a_2x + 3a_3x^2 + \cdots + na_nx^{n-1} \\
&= \sum_{k=1}^{n} ka_kx^{k-1} = \sum_{k=0}^{n-1}(k+1)a_{k+1}x^k.
\end{aligned}$$

Since p' is itself a polynomial, p is smooth, i.e., infinitely differentiable.

Example 8.23. Every rational function is smooth in its natural domain: If p and q are polynomials and $f(x) = p(x)/q(x)$, then f is differentiable for all real x with $q(x) \neq 0$, and f' is a rational function with the same natural domain as f. For example,

$$\begin{aligned} \frac{d}{dx}\frac{1}{1+x^2} &= -\frac{2x}{(1+x^2)^2}, \\ \frac{d}{dx}\frac{x}{1+x^2} &= \frac{(1+x^2) - x(2x)}{(1+x^2)^2} = \frac{1-x^2}{(1+x^2)^2}. \end{aligned}$$

Theorem 8.24 (The chain rule). *Let g and f be composable functions. If f is differentiable at x and g is differentiable at $y = f(x)$, then $g \circ f$ is differentiable at x, and*

$$(g \circ f)'(x) = g'\big(f(x)\big) \cdot f'(x) = g'(y)) \cdot f'(x).$$

Proof. By hypothesis,

$$\begin{aligned} f(x+h) &\approx f(x) + h\,f'(x) + o(h) \quad \text{for } h \approx 0, \\ g(y+k) &\approx g(y) + k\,g'(y) + o(k) \quad \text{for } k \approx 0. \end{aligned}$$

If we write $y = f(x)$ and $y + k = f(x+h)$, then

$$k = f(x+h) - f(x) \approx h\,f'(x) + o(h).$$

Thus $k \approx O(h)$ for $h \approx 0$, so $o(k) \approx o(h)$. Consequently,

$$\begin{aligned} (g \circ f)(x+h) - (g \circ f)(x) &= g(y+k) - g(y) \\ &\approx k\,g'(y) + o(k) \\ &\approx \big(h\,f'(x) + o(h)\big)\,g'(y) + o(h) \\ &\approx h\,g'\big(f(x)\big) \cdot f'(x) + o(h). \end{aligned}$$

By Proposition 8.11, $(g \circ f)'(x) = g'\big(f(x)\big) \cdot f'(x)$. ☐

Example 8.25. The derivatives of

$$f(x) = (1-x^2)^n, \quad \text{and} \qquad g(x) = \left(\frac{x}{1+x^2}\right)^n$$

(n an arbitrary integer) are

$$f'(x) = n(1-x^2)^{n-1}(-2x), \qquad g'(x) = n\left(\frac{x}{1+x^2}\right)^{n-1}\frac{1-x^2}{(1+x^2)^2}.$$

Remark 8.26. The preceding functions would be all but impossible to differentiate without the chain rule; the only recourse would be to multiply out and differentiate term by term.

Differentiability of Inverse Functions

Theorem 8.27. *Let X be an open interval in $\mathbf{R}$, and $f : X \to \mathbf{R}$ a strictly monotone function. If f is differentiable at some x_0 in X, then the inverse $g = f^{-1}$ is differentiable at $y_0 = f(x_0)$ if and only if $f'(x_0) \neq 0$, and in this event*

$$g'(y_0) = \frac{1}{f'(x_0)}.$$

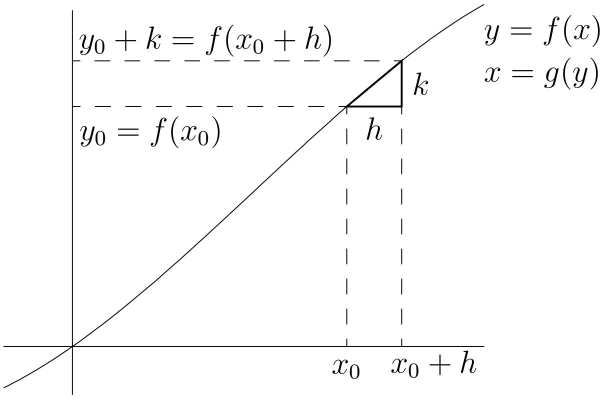


Figure 8.1: The difference quotient of an inverse function.

Proof. For every real number h such that $x_0 + h = x \in X$, we may write $y = f(x) = f(x_0 + h)$ and $k = y - y_0 = f(x_0 + h) - f(x_0)$, see Figure 8.1. Since f is injective, k is uniquely determined by h, and since f and f^{-1} are continuous, $o(h) \approx o(k)$ for $k \approx 0$. By algebra,

$$\begin{aligned}\Delta(f^{-1})(y_0, k) &= \frac{f^{-1}(y_0 + k) - f^{-1}(y_0)}{k} \\ &= \frac{h}{f(x_0 + h) - f(x_0)} = \frac{1}{\Delta f(x_0, h)}.\end{aligned}$$

If $f'(x_0) \neq 0$, then

$$(f^{-1})'(y_0) = \lim_{k \to 0} \Delta(f^{-1})(y_0, k) = \lim_{h \to 0} \frac{1}{\Delta f(x_0, h)} = \frac{1}{f'(x_0)}.$$

Conversely, $x = f^{-1}\big(f(x)\big)$ for all x in X, so if f^{-1} is differentiable at $y_0 = f(x_0)$, the chain rule implies

$$1 = (f^{-1})'(y_0) \cdot f'(x_0),$$

and therefore $f'(x_0) \neq 0$. ☐

Example 8.28. Let q be a positive integer, and let $f(x) = x^q$ for $x > 0$. The function f is increasing (hence invertible) and differentiable, with derivative $f'(x) = qx^{q-1} > 0$. The inverse function is the qth root function, $g(y) = y^{1/q}$. By Theorem 8.27, g is differentiable at $y = x^q$, and

$$g'(y) = \frac{1}{f'(x)} = \frac{1}{qx^{q-1}} = \frac{1}{q}y^{(1/q)-1}.$$

Combining with the chain rule, if p is an integer and if f is the power function $f(x) = x^{p/q} = (x^{1/q})^p$, then

$$f'(x) = p(x^{1/q})^{p-1} \cdot \frac{1}{q}x^{(1/q)-1} = \frac{p}{q}x^{(p/q)-1}.$$

8.3 The Mean Value Theorem

Theorem 8.29 (The Mean Value Theorem)**.** *Let $f : [a, b] \to \mathbf{R}$ be a continuous function that is differentiable on (a, b). Then there exists an x_0 in (a, b) such that*

$$f'(x_0) = \frac{f(b) - f(a)}{b - a}.$$

Remark 8.30. The superficial conclusion of the Mean Value Theorem has a plausible interpretation in terms of position and speed: If on a car trip you cover 60 miles in a certain one-hour period of time, then at some instant during that hour your speed must have been exactly 60 miles per hour.

However, the true power of the Mean Value Theorem arises because if x and y are arbitrary real numbers with $a \leq x < y \leq b$, we can apply the Mean Value Theorem to f on the interval $[x, y]$. That is, when traveling from Point A to Point B, your average speed over an *arbitrary* time interval is equal to your instantaneous speed at *some instant* during that interval.

Proof. Assume first that $f(a) = f(b)$. (This special case is known as *Rolle's theorem.*) By the Extreme Value Theorem, there exist points $x_{\min}$ and $x_{\max}$ in $[a, b]$ such that

$$f(x_{\min}) \leq f(x) \leq f(x_{\max}) \quad \text{for all } x \text{ in } [a, b].$$

Suppose at least one of $x_{\min}$ and $x_{\max}$ is in (a, b), and call it x_0. By Proposition 8.10,

$$f'(x_0) = 0 = \frac{f(b) - f(a)}{b - a},$$

as was to be shown.

The only other possibility is that each of the points $x_{\min}$ and $x_{\max}$ is an endpoint of $[a, b]$. But since $f(a) = f(b)$, this means f is a constant function, and then $f'(x_0) = 0$ for *every* point x_0 in (a, b).

This completes the proof of Rolle's theorem. To handle the general case, introduce the function $g : [a, b] \to \mathbf{R}$ defined by

$$g(x) = f(x) - \frac{f(b) - f(a)}{b - a}(x - a).$$

The function g is continuous on $[a, b]$, differentiable on (a, b), and a short calculation gives $g(a) = f(a) = g(b)$. By Rolle's theorem, there exists an x_0 in (a, b) such that $g'(x_0) = 0$. But

$$g'(x) = f'(x) - \frac{f(b) - f(a)}{b - a} \quad \text{for all } x \text{ in } (a, b). \qquad \square$$

The Identity Theorem

Theorem 8.31. *Let f and g be differentiable functions on some interval I. If $f'(x) = g'(x)$ for every interior point x of I, then there exists a real number C such that $f(x) = g(x) + C$ for all x in I.*

Proof. The function $h = f - g$ is differentiable, and by hypothesis, $h'(x) = f'(x) - g'(x) = 0$ for all x in the interior of I. It suffices to show h is a constant function. We prove the contrapositive.

If h is non-constant, then there exist numbers $a < b$ in I such that $h(a) \neq h(b)$. By the Mean Value Theorem applied to h on $[a, b]$, there exists an x_0 in (a, b) such that

$$h'(x_0) = \frac{h(b) - h(a)}{b - a} \neq 0. \qquad \square$$

Example 8.32. The signum function $\operatorname{sgn} x = x/|x|$, defined for all $x \neq 0$, has derivative identically 0, but is not constant. This example shows the importance of assuming the domain is an *interval.*

Monotonicity

Theorem 8.33. *If $f : [a, b] \to \mathbf{R}$ is continuous, differentiable on (a, b), and if $f'(z) > 0$ for all z in (a, b), then f is strictly increasing in $[a, b]$, and f is a bijection to the closed interval $\big[f(a), f(b)\big]$.*

Proof. Let x and y be arbitrary numbers such that $a \leq x < y \leq b$. By the Mean Value Theorem applied to f on $[x, y]$, there is an x_0 in (x, y) such that

$$\frac{f(y) - f(x)}{y - x} = f'(x_0).$$

By hypothesis, $f'(x_0) > 0$, and since $x < y$, it follows that $f(x) < f(y)$. Since $x < y$ were arbitrary, f is strictly increasing.

In particular, f is injective, and the image of f is contained in $\big[f(a), f(b)\big]$: If $a \leq x \leq b$, then $f(a) \leq f(x) \leq f(b)$. Conversely, if $f(a) < y < f(b)$, the Intermediate Value Theorem guarantees there exists an x in (a, b) such that $y = f(x)$; that is, $\big[f(a), f(b)\big]$ is contained in the image of f. $\square$

Remark 8.34. An entirely analogous argument shows that if f is continuous on $[a, b]$ and $f'(x) < 0$ for all x in (a, b), then f is decreasing on $[a, b]$, and the image of f is $\big[f(b), f(a)\big]$.

Example 8.35. The polynomial function $f(x) = x^3 - 3x$ is differentiable on $\mathbf{R}$, and $f'(x) = 3x^2 - 3 = 3(x - 1)(x + 1)$. Since $f'(x) > 0$ if $x < -1$, f is strictly increasing on the *closed* interval $(-\infty, -1]$. Further, the image of f on this interval is the interval $(-\infty, 2]$, since $f(x) \to -\infty$ as $x \to -\infty$ and $f(-1) = 2$, Figure 8.2.

Since $f'(x) < 0$ if $-1 < x < 1$, f is strictly decreasing on the closed interval $[-1, 1]$, and f maps this interval to $[-2, 2]$.

Similarly, f is strictly increasing on $[1, \infty)$, and f maps this interval to $[-2, \infty)$.

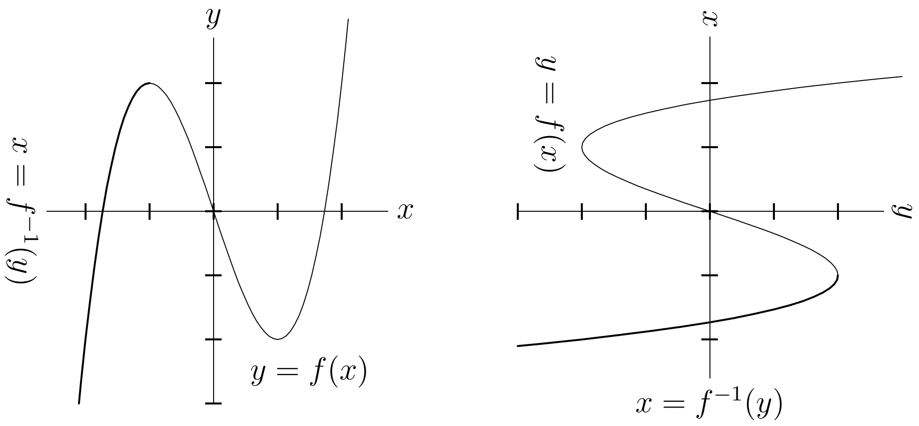


Figure 8.2: Branches of f^{-1} for $f(x) = x^3 - 3x$.

It follows from this analysis that $f : \mathbf{R} \to \mathbf{R}$ is surjective; the equation $y = x^3 - 3x$ has at least one solution x for every real y. In fact, we have shown more. If $|y| < 2$, the equation $y = x^3 - 3x$ has precisely three solutions: one less than -1, one between -1 and 1, and one greater than 1. If $y = \pm 2$, there are precisely two solutions (one being ∓ 1). If $|y| > 2$, there is exactly one solution of $y = x^3 - 3x$.

We have also demonstrated the existence of a unique continuous function $g : (-\infty, 2] \to (-\infty, -1]$ satisfying $g(x^3 - 3x) = x$ for all $x \leq -1$ and $g(y)^3 - 3g(y) = y$ for all $y \leq 2$. The function g, a so-called *branch of* f^{-1}, is differentiable in the open interval $(-\infty, 2)$, and

$$g'(y) = \frac{1}{f'(x)} = \frac{1}{3(x^2 - 1)} = \frac{1}{3(g(y)^2 - 1)}.$$

Similarly, there is a continuous branch of f^{-1} from $[-2, 2]$ to $[-1, 1]$, and a continuous branch of f^{-1} from $[-2, \infty)$ to $[1, \infty)$.

Convexity

Definition 8.36. A function $f : [a, b] \to \mathbf{R}$ is *convex* if, for all numbers x and y with $a \leq x < y \leq b$, we have

$$f(z) \leq f(x) + \frac{f(y) - f(x)}{y - x}(z - x) \quad \text{for all } z \text{ with } x < z < y.$$

If the preceding inequality is strict, we say f is *strictly convex.*

Remark 8.37. Geometrically, f is convex if the graph of f lies on or below *every* secant line of the graph.

We say f is *concave* if $-f$ is convex, i.e., if the graph of f lies on or above every secant line of the graph.

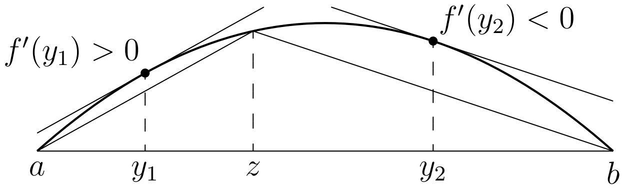
Definition 8.38. Let f be continuous in some neighborhood of a real number x_0. We say the point $\big(x_0, f(x_0)\big)$ on the graph of f is an *inflection point* of the graph $y = f(x)$ if "the concavity of f changes at x_0", namely, if there exists an open ball $B_r(x_0)$ such that f is strictly convex on $(x_0 - r, x_0)$ and strictly concave on $(x_0, x_0 + r)$, or else is strictly concave on $(x_0 - r, x_0)$ and strictly convex on $(x_0, x_0 + r)$.

Lemma 8.39. *If f is continuous on $[a, b]$, vanishes at the endpoints, and is twice-differentiable on (a, b), with $f''(x) \geq 0$ for all x in (a, b), then $f(z) \leq 0$ for all z in (a, b).*

Proof. Assume contrapositively that there exists a z in (a, b) with $f(z) > 0$. It suffices to prove $f''(x) < 0$ for some x in (a, b). By the Mean Value Theorem applied to f on $[a, z]$, there exists a y_1 in (a, z) such that

$$f'(y_1) = \frac{f(z) - f(a)}{z - a} = \frac{f(z)}{z - a} > 0.$$

Similarly, there is a point y_2 in (z, b) with $f'(y_2) < 0$.



Applying the Mean Value Theorem to f' on $[y_1, y_2]$, there is an x in (y_1, y_2) such that

$$f''(x) = \frac{f'(y_2) - f'(y_1)}{y_2 - y_1} < 0. \qquad \square$$

Theorem 8.40. *If f is twice differentiable on $[a, b]$, and $f''(z) \geq 0$ for all z in (a, b), then f is convex on $[a, b]$.*

Proof. Fix arbitrary numbers $x < y$ in $[a, b]$, and consider the function

$$g(z) = f(z) - \left(f(x) + \frac{f(y) - f(x)}{y - x}(z - x)\right).$$

By construction, $g(x) = g(y) = 0$ and $g'' = f'' \geq 0$. Lemma 8.39 implies $g(z) \leq 0$ on $[x, y]$. Since $x < y$ were arbitrary, f is convex. $\square$

Remark 8.41. The absolute value function is convex on $\mathbf{R}$, but is not even once differentiable. That is, convexity does not imply $f'' \geq 0$, see also Exercise 8.18.

Corollary 8.42. *If f is continuous on $[a, b]$, twice-differentiable with $f'' \geq 0$ on (a, b), and if $f'' = 0$ only at isolated points in (a, b), then f is strictly convex on $[a, b]$.*

Proof. Fix $x < y$ in $[a, b]$. In the notation above, it suffices to prove $g(z) < 0$ (strict inequality) for $x < z < y$. By Theorem 8.40, $g \leq 0$ in (x, y). However, g cannot be identically zero, because $g'' = 0$ only at isolated points. Thus, $g(z_0) < 0$ for some z_0 in (x, y).

Since g is convex by the theorem,

$$g(z) \leq g(x) + \frac{g(z_0) - g(x)}{z_0 - x}(z - x) = \frac{g(z_0)}{z_0 - x}(z - x) < 0$$

for $x < z \leq z_0$. A similar argument shows $g(z) < 0$ if $z_0 < z < y$. $\square$

Example 8.43. The function $f(x) = x^3$ is twice-differentiable, and $f''(x) = 6x$. By Corollary 8.42, the graph of f has an inflection point at the origin.

Example 8.44. The function $f(x) = x^4$ is twice-differentiable, and $f''(x) = 12x^2 \geq 0$ for all real x, with equality only if $x = 0$. By Corollary 8.42, f is strictly convex on $\mathbf{R}$.

Exercises

Exercise 8.1. Find all the derivatives of the following functions:
(a) $s(x) = x - (x^3/3!)$. (b) $c(x) = 1 - (x^2/2!) + (x^4/4!)$.
(c) $f(x) = 1 + x + (x^2/2!) + (x^3/3!)$.

Exercise 8.2. Let $n > 0$ be an integer, and $f(x) = x^n$. Find all the derivatives of f, expressing them in terms of factorials and powers of x. In particular, show that $f^{(n)}(x) = n!$, and that $f^{(k)}(x) = 0$ for $k > n$. Suggestion: Calculate derivatives "by hand" until a pattern emerges. Then use induction to prove this pattern is correct.

Exercise 8.3. Let $f(x) = (1+x)^{1/2}$ for $x > -1$. Find all the derivatives of f. Express $f^{(k)}(0)$ in terms of factorials.

Exercise 8.4. In each part, let $f(x) = (1 - x^2)^2$.

(a) Find the intervals of monotonicity and convexity of f, the zeros of f, and any local extrema and inflection points. Use this information to sketch the graph $y = f(x)$.

(b) Use the techniques of Example 8.35 to find corresponding branches of f^{-1}. Be specific about the domain and image of each branch, and whether each branch is an increasing or decreasing function.

(c) Solve the equation $y = f(x)$ for x in terms of y. Match each formula you find with a branch of f^{-1} found in (a).

Exercise 8.5. In each part, let $f(x) = 1/(1+x^2)$.

(a) Find the intervals of monotonicity and convexity of f, the zeros of f, and any local extrema and inflection points. Use this information to sketch the graph $y = f(x)$.

(b) Use the techniques of Example 8.35 to find corresponding branches of f^{-1}. Be specific about the domain and image of each branch, and whether each branch is an increasing or decreasing function.

(c) Solve the equation $y = f(x)$ for x in terms of y. Match each formula you find with a branch of f^{-1} found in (a).

Exercise 8.6. In each part, let $f(x) = x/(1+x^2)$.

(a) Find the intervals of monotonicity and convexity of f, the zeros of f, and any local extrema and inflection points. Use this information to sketch the graph $y = f(x)$.

(b) Use the techniques of Example 8.35 to find corresponding branches of f^{-1}. Be specific about the domain and image of each branch, and whether each branch is an increasing or decreasing function.

(c) Solve the equation $y = f(x)$ for x in terms of y. Match each formula you find with a branch of f^{-1} found in (a).

Exercise 8.7. In each part, let $f(x) = x/(1-x^2)$ for $x \neq \pm 1$.

(a) Find the intervals of monotonicity and convexity of f, the zeros of f, and any local extrema and inflection points. Use this information to sketch the graph $y = f(x)$.

(b) Use the techniques of Example 8.35 to find corresponding branches of f^{-1}. Be specific about the domain and image of each branch, and whether each branch is an increasing or decreasing function.

(c) Solve the equation $y = f(x)$ for x in terms of y. Match each formula you find with a branch of f^{-1} found in (a).

Exercise 8.8. In each part, let $f(x) = 1/\sqrt{1+x^2}$. Find the intervals of monotonicity and convexity of f, the zeros of f, and any local extrema and inflection points. Sketch the graph $y = f(x)$.

Exercise 8.9. Let $f(x) = (1-x)x^{2/3}$. Find the intervals of monotonicity and convexity of f, the zeros of f, and any local extrema and inflection points. Use this information to sketch the graph $y = f(x)$.

Exercise 8.10. Let m and n be positive integers, and let f be the polynomial function $f(x) = (1-x)^n(1+x)^m$. Find the intervals of monotonicity of f, the zeros of f, and any local extrema. Use this information to sketch the graph $y = f(x)$.

Exercise 8.11. Match each Leibniz notation formula with the corresponding proposition or theorem from this chapter, and specify the appropriate functional relationships.

(a) $\dfrac{d(cy+z)}{dx} = c\,\dfrac{dy}{dx} + \dfrac{dz}{dx}$, (c) $\dfrac{dz}{dx} = \dfrac{dz}{dy}\,\dfrac{dy}{dx}$,

(b) $\dfrac{d(uv)}{dx} = u\,\dfrac{dv}{dx} + v\,\dfrac{du}{dx}$, (d) $\dfrac{dx}{dy} = \dfrac{1}{dy/dx}$.

Exercise 8.12. Let f be a twice continuously differentiable function; that is, f'' exists and is continuous. Assume $f(x_0) > 0$, and define $g(x) = 1/f(x)$ provided $f(x) \neq 0$.

(a) Find a formula for g'' in a neighborhood of x_0. Your formula should depend only on the values of f and its first two derivatives at x_0.

(b) If $f''(x_0) > 0$ what (if anything) can you say about the sign of $g''(x_0)$? What if $f''(x_0) < 0$?

Exercise 8.13. Let f be twice continuously differentiable. Assume $f'(x_0) \neq 0$, so that f is invertible near x_0, and set $y_0 = f(x_0)$.

(a) If g is a branch of f^{-1} satisfying $g(y_0) = x_0$, prove that g is twice differentiable at y_0, and find an expression for g'' in a neighborhood of y_0. Your formula should depend only on the values of f and its first two derivatives at x_0.

(b) If $f''(x_0) > 0$ what (if anything) can you say about the sign of $g''(y_0)$? What if $f''(x_0) < 0$?

Exercise 8.14. Let f be a differentiable, convex function on $[a, b]$, and let P be an arbitrary partition. Prove that

$$\text{MID}(f, P) \le \int_a^b f \le \text{TRAP}(f, P),$$

(see Exercise 7.10 for the definitions) and illustrate with a sketch.

Exercise 8.15. Let $\psi : \mathbf{R} \to \mathbf{R}$ be a smooth, non-constant, 1-periodic function. (That is, ψ is infinitely-differentiable, and $\psi(x+1) = \psi(x)$ for all real x. We prove in Chapter 11 that such functions exist, for example, $\psi(x) = \sin(2\pi x)$.) Define $f(x) = x^2\psi(1/x)$ if $x \neq 0$, and $f(0) = 0$.

(a) Prove that ψ is bounded, i.e., there exists a real number $M > 0$ such that $|\psi(x)| \leq M$ for all real x.

(b) For $x \neq 0$, calculate $f'(x)$ in terms of $\psi(x)$ and $\psi'(x)$. Prove that f' is bounded.

(c) Prove $f'(0)$ exists, but f' is discontinuous at 0.

Exercise 8.16. With notation as in the preceding exercise, define $f(x) = x^2\psi(1/x^2)$ if $x \neq 0$, and $f(0) = 0$. Prove that f is differentiable on $\mathbf{R}$, but f' is unbounded in every neighborhood of 0.

Exercise 8.17. Let I be an interval and $f : I \to \mathbf{R}$ a non-decreasing (hence integrable) function. Assume $a \in I$, and let F be the definite integral of f from a. This exercise establishes that F is convex, namely

$$F(z) \leq F(x) + \frac{z-x}{y-x}\big(F(y) - F(x)\big) \quad \text{for all } x < z < y \text{ in } I.$$

(a) Show that convexity of F is equivalent to

$$\frac{1}{z-x}\int_x^z f \leq \frac{1}{y-z}\int_z^y f \quad \text{for all } x < z < y \text{ in } I,$$

and use monotonicity of f to establish this condition.

(b) Prove f is strictly increasing if and only if F is strictly convex. Caution: f need not be continuous on any interval.

Exercise 8.18. Let f be the function constructed in Exercise 6.16, and let F be the definite integral of f from 0. Prove that F is continuous, increasing, and strictly convex, but fails to be differentiable at every rational number. Suggestion: Use Exercise 8.17.

Exercise 8.19. Let f be a differentiable function, ℓ a non-vertical line through an arbitrary point $\big(x_0, f(x_0)\big)$, and $\varepsilon > 0$ arbitrary. This problem constructs a differentiable function g that (i) differs from f everywhere by at most ε, and (ii) has ℓ as a tangent line.

(a) Define $\phi(x) = x/(1+x^2)$. Calculate ϕ', and determine the absolute maximum and minimum values of ϕ.

(b) Let m be an arbitrary real number, and define $\phi_m(x) = \phi(mx)$. Show that the line $y = mx$ is tangent to the graph of ϕ_m at $x = 0$.

(c) Let $n \geq 0$ be an integer, and define $\psi_n(x) = \frac{1}{n}\phi_m(nx)$. Prove that (i) The line $y = mx$ is tangent to the graph of ψ_n at the origin, and (ii) For sufficiently large n, $|\psi_n(x)| < \varepsilon$ for all real x.

(d) Let $f : \mathbf{R} \to \mathbf{R}$ be differentiable, and ℓ a non-vertical line through an arbitrary point $\big(x_0, f(x_0)\big) = (x_0, y_0)$ on the graph of f.

By scaling and shifting the function ψ constructed in part (c), prove that for every $\varepsilon > 0$, there exists a differentiable function g, defined on the same set as f, such that (i) $g(x_0) = f(x_0)$, (ii) The line ℓ is tangent to the graph of g at (x_0, y_0), and (iii) $|g(x) - f(x)| < \varepsilon$ for all x in the domain of f.

In words, by perturbing f by an arbitrarily small amount in the vertical direction, you can make the graph tangent to an arbitrary line. Thus, we have no visual basis for saying any particular line is mathematically tangent to any particular graph.

Exercise 8.20. (Darboux's theorem, a.k.a. the Intermediate Value Property for Derivatives) Let $f : I \to \mathbf{R}$ be differentiable, and assume $a < b$ are points of I satisfying $f'(a) < f'(b)$.

Prove that if k is a real number such that $f'(a) < k < f'(b)$, there exists an x in (a, b) such that $f'(x) = k$.
Caution: We do not know f' is continuous. Instead, apply the Extreme Value Theorem to the differentiable function $g(x) = f(x) - kx$ on the interval $[a, b]$, and use the assumptions on the endpoint derivatives to show that neither endpoint is an extremum.

Exercise 8.21. Let f be differentiable on some open ball $B_r(x_0)$. Prove that if the one-sided limits $\lim(f', x_0^-)$ and $\lim(f', x_0^+)$ exist, then they are both equal to $f'(x_0)$.
Hint: Show that if a one-sided limit exists but is unequal to $f'(x_0)$, then Darboux's theorem (preceding question) is false.

In words, a derivative cannot have a jump discontinuity.

Exercise 8.22. Let f be continuous on some open ball $B_r(x_0)$, differentiable on the deleted ball $B_r^\times(x_0)$, and assume the one-sided limits $\lim(f', x_0^-)$ and $\lim(f', x_0^+)$ exist and are equal. Prove f is differentiable at x_0, and $f'(x_0) = \lim(f', x_0^-) = \lim(f', x_0^+)$.
Hint: Apply the Mean Value Theorem to intervals contained in $B_r(x_0)$ and having x_0 as an endpoint.

Exercise 8.23. Find all values of a_1, a_2, b_1, b_2 such that the function

$$f(x) = \begin{cases} a_1 + \dfrac{a_2}{x} & x < -1, \\ 4 - x^2 & -1 \leq x \leq 1, \\ b_1 + \dfrac{b_2}{x^2} & 1 < x \end{cases}$$

is differentiable on $\mathbf{R}$.

Exercise 8.24. This exercise outlines *l'Hôpital's rule*, a computational procedure for evaluating certain indeterminate limits of the form 0/0. The technical tool is the *Cauchy Mean Value Theorem*, given in part (a). Throughout, f and g denote real-valued functions.

(a) Assume f, g are continuous on some interval $[a, b]$ and differentiable on (a, b). Prove there exists a point z in (a, b) such that

$$f'(z)\big(g(b) - g(a)\big) = g'(z)\big(f(b) - f(a)\big).$$

Hint: Apply Rolle's Theorem to the function $h : [a, b] \to \mathbf{R}$ defined by $h(x) = f(x)\big(g(b) - g(a)\big) - g(x)\big(f(b) - f(a)\big)$.

(b) Suppose f and g are differentiable in some deleted neighborhood X of x_0, that g is non-vanishing in this neighborhood, and that $\lim(f, x_0) = \lim(g, x_0) = 0$.

Prove that if $\lim(f'/g', x_0) = \ell$, then $\lim(f/g, x_0) = \ell$. (In English: When attempting to evaluate a limit of a quotient, if the answer is formally 0/0, try differentiating the numerator and denominator and re-evaluating. If the limit is ℓ, then the original limit was also ℓ.)

Hint: Setting $f(x_0) = 0$ and $g(x_0) = 0$ extends f and g continuously. If $\delta > 0$ is chosen so that $B_\delta^\times(x_0) \subseteq X$, use the Cauchy Mean Value Theorem to write

$$\frac{f(x_0 + \delta)}{g(x_0 + \delta)} = \frac{f'(z)}{g'(z)}$$

for some z between x_0 and $x_0 + \delta$, then take the limit as $\delta \to 0$.

(c) Suppose f and g are differentiable and g is non-vanishing on some interval (R, ∞), and that $\lim(f, \infty) = 0$, $\lim(g, \infty) = 0$. Prove that if $\lim(f'/g', \infty) = \ell$, then $\lim(f/g, \infty) = \ell$.

Hint: For $x > R$, define $F(x) = f(1/x^2)$ and $G(x) = g(1/x^2)$, then show part (b) can be applied at 0.

Exercise 8.25. In each part, let r be a rational number and $c > 0$ real. Use l'Hôpital's rule (Exercise 8.24) to evaluate the stated limit.

(a) $\lim\limits_{x\to c} \dfrac{x^r - c^r}{x - c}$.

If r is a positive integer, check that your result agrees with the geometric series formula.

(b) $\lim\limits_{x\to c} \dfrac{(x^2 + c^2)^r - (2c^2)^r}{x - c}$.

Exercise 8.26. Let f be defined in some neighborhood of x_0.

(a) Assuming f is differentiable at x_0, evaluate

$$\lim_{h\to 0} \frac{f(x_0 + h) - f(x_0 - h)}{2h}.$$

(b) If the limit in part (a) exists at x_0, does it follow that f is differentiable at x_0? Continuous?

Exercise 8.27. Let f be twice-differentiable in a neighborhood x_0. Evaluate

$$\lim_{h\to 0} \frac{f(x_0 + h) + f(x_0 - h) - 2f(x_0)}{h^2}.$$

Why do we not use this limit to define f''?

Chapter 9

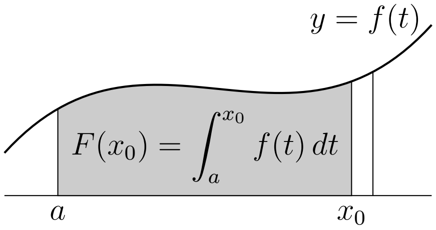
The Fundamental Theorems

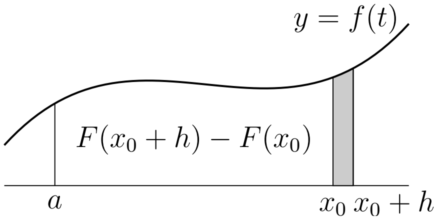
Theorem 9.1 (The first fundamental theorem of calculus)**.** *Let f be an integrable function on $[a,b]$, and let F be the definite integral*

$$F(x) = \int_a^x f, \quad \textit{defined for } x \textit{ in } [a,b].$$

If f is continuous at x_0 in (a,b), then F is differentiable at x_0, and $F'(x_0) = f(x_0)$. In Leibniz notation,

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x) \quad \textit{if } f \textit{ is continuous at } x.$$





Proof. If $x_0 + h$ lies in (a,b), then

$$\begin{aligned}\Delta F(x_0,h) &= \frac{F(x_0+h) - F(x_0)}{h} \\ &= \frac{1}{h}\left[\int_a^{x_0+h} f - \int_a^{x_0} f\right] = \frac{1}{h}\int_{x_0}^{x_0+h} f.\end{aligned}$$

Since f is continuous at x_0, $f(x) \approx f(x_0) + o(1)$ for $x \approx x_0$, so

$$\Delta F(x_0,h) \approx \frac{1}{h}\int_{x_0}^{x_0+h} \big(f(x_0) + o(1)\big) \approx f(x_0) + o(1) \quad \text{for } h \approx 0.$$

Letting $h \to 0$ shows $F'(x_0) = f(x_0)$. □

Theorem 9.2 (The second fundamental theorem of calculus). *Let F be continuously differentiable on some open interval I. If $a \in I$, then*

$$\int_a^x F'(t)\,dt = F(x) - F(a) \quad \textit{for all } x \textit{ in } I.$$

Proof. Consider the definite integral

$$G(x) = \int_a^x F'(t)\,dt.$$

By the First Fundamental Theorem, G is differentiable in I, and $G'(x) = F'(x)$ for all x in I. By the identity theorem, there exists a real number C such that $G(x) = F(x) + C$. To evaluate C, set $x = a$: $0 = G(a) = F(a) + C$, so $C = -F(a)$, or $G(x) = F(x) - F(a)$. □

9.1 Integration by Parts and Substitution

Loosely, the fundamental theorems assert that integration and differentiation are inverse operations. Consequently, the product rule and chain rule for differentiation each correspond to an integral formula.

Proposition 9.3 (Integration by Parts). *If u and v are continuously differentiable on $[a, b]$, then*

$$\int_a^b uv' = uv\Big|_a^b - \int_a^b vu'.$$

Proof. By the product rule for derivatives, $(uv)' = u'v + uv'$, or

$$uv' = (uv)' - vu'.$$

The proposition follows from Theorem 9.2 by integrating each side over $[a, b]$. □

Theorem 7.22 has a useful, far-reaching generalization.

Theorem 9.4 (Change of variables). *Let $a < b$ be real numbers, τ a continuously differentiable function on $[a, b]$, and f a continuous function whose domain contains the image of τ. Then*

$$\int_{\tau(a)}^{\tau(b)} f = \int_a^b (f \circ \tau) \cdot \tau'.$$

Remark 9.5. In Leibniz notation, put $t = \tau(s)$, so that $dt = \tau'(s)\,ds$. The conclusion of the theorem reads

$$\int_{\tau(a)}^{\tau(b)} f(t)\,dt = \int_a^b f\bigl(\tau(s)\bigr) \cdot \tau'(s)\,ds.$$

Proof. Consider the continuously differentiable functions

$$G(x) = \int_{\tau(a)}^{x} f, \qquad F(s) = (G \circ \tau)(s) = \int_{\tau(a)}^{\tau(s)} f.$$

By the chain rule and the First Fundamental Theorem,

$$F' = (G' \circ \tau) \cdot \tau' = (f \circ \tau) \cdot \tau'.$$

By the Second Fundamental Theorem,

$$\int_{\tau(a)}^{\tau(b)} f = F(b) - F(a) = \int_a^b F' = \int_a^b (f \circ \tau) \cdot \tau'. \qquad \square$$

9.2 Taylor's Theorem

Definition 9.6. Let f be n times differentiable in some neighborhood of x_0. Define the nth degree *Taylor polynomial* (of f at x_0) by

$$p_n(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n,$$

and the nth degree *remainder* by $R_n(x) = f(x) - p_n(x)$.

Theorem 9.7. *Let n be a non-negative integer. If f is $(n+1)$ times continuously differentiable in some open interval X containing x_0, i.e., is of class $\mathscr{C}^{n+1}$ near x_0, then for each x in X:*

(i) (*Integral form of the remainder*)

$$R_n(x) = \frac{1}{n!} \int_{x_0}^{x} f^{(n+1)}(t)(x-t)^n\,dt.$$

(ii) (*Lagrange form of the remainder*) *There is a z_{n+1} between x_0 and x such that*

$$R_n(x) = \frac{f^{(n+1)}(z_{n+1})}{(n+1)!}(x-x_0)^{n+1}.$$

Remark 9.8. Philosophically, Taylor's theorem says that on an interval about a point x_0, a function of class $\mathscr{C}^{n+1}$ "behaves like a polynomial of degree n up to order $(n+1)$. Particularly, if f is smooth, then for each $n \geq 0$, the nth degree Taylor polynomial $p_n(x)$ approximates $f(x)$ to degree $(n+1)$, i.e.,

$$f(x) \approx p_n(x) + O(x - x_0)^{n+1} \quad \text{for } x \approx x_0,$$

with the constant in O bounded by the maximum of $|f^{(n+1)}|$ on X. When $x \approx x_0$, a larger degree corresponds to a better approximation.

Proof. When $n = 0$, the integral form of the remainder comes from Theorem 9.2:

$$f(x) - p_0(x) = f(x) - f(x_0) = \int_{x_0}^{x} f'(t)\,dt.$$

Assume inductively that

$$R_n(x) = \frac{1}{n!}\int_{x_0}^{x} f^{(n+1)}(t)(x-t)^n\,dt$$

for some $n \geq 0$. Integrating by parts with

$$\begin{aligned} u(t) &= f^{(n+1)}(t), & v(t) &= -\frac{1}{(n+1)!}(x-t)^{n+1}, \\ u'(t) &= f^{(n+2)}(t), & v'(t) &= \frac{1}{n!}(x-t)^n, \end{aligned}$$

gives

$$\begin{aligned} R_n(t) &= -\frac{f^{(n+1)}(t)}{(n+1)!}(x-t)^{n+1}\bigg|_{t=x_0}^{t=x} + \frac{1}{(n+1)!}\int_{x_0}^{x} f^{(n+2)}(t)(x-t)^{n+1}\,dt \\ &= \frac{f^{(n+1)}(x_0)}{(n+1)!}(x-x_0)^{n+1} + \frac{1}{(n+1)!}\int_{x_0}^{x} f^{(n+2)}(t)(x-t)^{n+1}\,dt, \end{aligned}$$

or

$$f(x) = p_n(x) + R_n(x) = p_{n+1}(x) + \frac{1}{(n+1)!}\int_{x_0}^{x} f^{(n+2)}(t)(x-t)^{n+1}\,dt,$$

the asserted form of $R_{n+1}(x)$.

To establish the Lagrange form of the remainder, note that $f^{(n+1)}$ is continuous on the interval between x_0 and x, so $f^{(n+1)}$ achieves an absolute minimum value m and an absolute maximum value M in this interval. By monotonicity of the integral,

$$\frac{1}{n!}\int_{x_0}^{x} m(x-t)^n\,dt \leq \frac{1}{n!}\int_{x_0}^{x} f^{(n+1)}(t)(x-t)^n\,dt \leq \frac{1}{n!}\int_{x_0}^{x} M(x-t)^n\,dt,$$

or

$$m\frac{(x-x_0)^{n+1}}{(n+1)!} \leq R_n(x) \leq M\frac{(x-x_0)^{n+1}}{(n+1)!}.$$

Again since $f^{(n+1)}$ is continuous, the Intermediate Value Theorem guarantees there exists a z_{n+1} between x_0 and x such that

$$R_n(x) = \frac{f^{(n+1)}(z_{n+1})}{(n+1)!}(x-x_0)^{n+1}. \qquad \square$$

Convexity and the Second Derivative

In calculus, a function f is said to be *convex* if the graph of f "lies above each tangent line". Books often assert that f is convex provided $f'' > 0$. Taylor's theorem may be used to establish this fact. Note, however, that the secant criterion for convexity in Definition 8.36 makes no assumption of differentiability (much less continuity of the second derivative), and is therefore both simpler and more general than the usual calculus definition.

Proposition 9.9. *If f'' is continuous and $f''(x_0) > 0$, then on some open ball $B_r(x_0)$, the graph of f lies above each of its tangent lines.*

Proof. Since $f''(x_0) > 0$ and f'' is continuous, Corollary 6.17 guarantees there is an $r > 0$ such that if $|x - x_0| < r$, then $f''(x) > f''(x_0)/2 > 0$.

Let x_1 be an arbitrary point of $B_r(x_0)$. By Taylor's theorem at x_1, using the integral form of the remainder

$$f(x) = p_1(x) + R_1(x) = f(x_1) + f'(x_1)(x-x_1) + \frac{1}{2!}\int_{x_1}^{x} f''(t)(x-t)\,dt.$$

But the graph $y = p_1(x)$ is the tangent line to the graph $y = f(x)$ at x_1, and if $|x - x_0| < r$, i.e., if $x \in B_r(x_0)$, the integral is positive except at x_1. (If $x < x_1$, the integrand is negative but the interval of integration is oriented from right to left.) That is, on the open interval $B_r(x_0)$ the graph of f lies above the tangent line at x_1. $\square$

9.3 Differentiation of Power Series

Theorem 9.10. *Suppose*

$$\begin{aligned} f(x) &= \sum_{k=0}^{\infty} a_k (x - x_0)^k \\ &= a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + a_3(x - x_0)^3 + \dots \end{aligned}$$

is a convergent power series with radius $R > 0$. The real-analytic function f is differentiable at x for $|x - x_0| < R$, and

$$\begin{aligned} f'(x) &= \sum_{k=1}^{\infty} k a_k (x - x_0)^{k-1} = \sum_{k=0}^{\infty} (k+1) a_{k+1} (x - x_0)^k \\ &= a_1 + 2a_2(x - x_0) + 3a_3(x - x_0)^2 + 4a_4(x - x_0)^3 + \dots . \end{aligned}$$

Remark 9.11. Computationally, a convergent power series can be differentiated term by term in its interval of convergence, just as if it were a polynomial.

The theorem guarantees the termwise derived series has the same radius of convergence as the original power series. This has an important "bootstrapping" consequence: Since f' is real-analytic on $(-R, R)$, the *second* derivative f'' can be represented by a convergent power series on $(-R, R)$, obtained by differentiating the series for f' term by term, and so forth. That is, a real-analytic function is *infinitely differentiable.*

Proof. As usual, assume $x_0 = 0$. Throughout the proof, let

$$\begin{aligned} f_n(x) &= \sum_{k=0}^{n} a_k x^k, && \text{the polynomial approximators of } f(x); \\ g(x) &= \sum_{k=1}^{\infty} k a_k x^{k-1}, && \text{the termwise derived series;} \\ f_n'(x) &= \sum_{k=1}^{n} k a_k x^{k-1}, && \text{the polynomial approximators of } g(x). \end{aligned}$$

Our first task is to prove that the series $g(x)$ converges absolutely for all x in $(-R, R)$. By a now-familiar trick, we may write $|x| = r\rho$ with $|x| < |r| < R$ and $\rho < 1$. The terms of $g(x)$ are therefore bounded in absolute value by

$$|k a_k x^{k-1}| \leq |a_k r^{k-1}| \cdot |k \rho^{k-1}|.$$

By Corollary 4.79, $(k\rho^{k-1}) \to 0$, so the second factor is bounded. The first factor is, aside from a multiplied factor of r, the general term of the series for $f(r)$, which is absolutely convergent by hypothesis. The derived series $g(x)$ is therefore absolutely convergent on $(-R, R)$, and consequently may be integrated term by term:

$$\begin{aligned} \int_0^x g(t)\,dt &= \int_0^x \lim_{n\to\infty} f_n'(t)\,dt \\ &= \lim_{n\to\infty} \int_0^x f_n'(t)\,dt && \text{Theorem 7.41} \\ &= \lim_{n\to\infty} f_n(x) - f_n(0) && \text{Theorem 9.2} \\ &= f(x) - f(0). \end{aligned}$$

By Theorem 9.1,

$$f'(x) = \frac{d}{dx} \int_0^x g(t)\,dt = g(x). \qquad \square$$

9.4 Improper Integrals

Definition 9.12. An integral $\int_a^b f$ is called a *basic improper integral* if one of the following holds:

(i) Exactly one endpoint a or b is infinite, and f is bounded.

(ii) Both endpoints are finite and f is unbounded, but only in a neighborhood of one endpoint.

An integral is *improper* if the interval of integration can be divided into finitely many subintervals, on each of which the integral is a basic improper integral.

Example 9.13. Let $p > 0$ be real. The following are basic improper integrals:

$$\int_0^1 \frac{dx}{x^p}, \qquad \int_{-3}^0 \frac{dx}{|x|^p}, \qquad \int_1^\infty \frac{dx}{x^p}.$$

The following are improper, and are split into basic improper pieces:

$$\int_0^\infty \frac{dx}{x^p} = \int_0^1 \frac{dx}{x^p} + \int_1^\infty \frac{dx}{x^p}, \quad \int_{-1}^\infty \frac{dx}{|x|^p} = \int_{-1}^0 \frac{dx}{|x|^p} + \int_0^1 \frac{dx}{|x|^p} + \int_1^\infty \frac{dx}{|x|^p}.$$

Definition 9.14. (i) A basic improper integral $\int_a^\infty f$ is said to *converge* to L if

$$L = \lim_{b\to\infty} \int_a^b f.$$

Basic improper integrals $\int_{-\infty}^b f$ are handled similarly.

A basic improper integral $\int_a^b f$ with f unbounded near b is said to *converge* to L if

$$L = \lim_{x\to b^-} \int_a^x f.$$

An analogous definition is made for $\int_a^b$ if f is unbounded near a.

A basic improper integral that does not converge is said to *diverge*.

An improper integral is said to *converge to L* if the integral can be broken into finitely many basic sub-integrals, each of which converges, and the sum of the sub-integrals is L. Otherwise, the improper integral is said to *diverge*.

Proposition 9.15. *Let $p > 0$, and define $f(x) = x^p$ if $x > 0$, $f(0) = 0$.*

(i) *The improper integral* $\displaystyle\int_0^1 \frac{dx}{x^p}$ *converges if and only if $p < 1$, and the value is $1/(1-p)$.*

(ii) *The improper integral* $\displaystyle\int_1^\infty \frac{dx}{x^p}$ *converges if and only if $p > 1$, and the value is $1/(p-1)$.*

Remark 9.16. We have not yet defined exponentiation with irrational exponents, so strictly speaking the proof below should be read as applying only to rational exponents p. In light of Theorem 10.7 below, the proof below goes through without modification for arbitrary real $p > 0$.

Proof. First suppose $p = 1$. The definite integral of $1/x$ from 1 is the natural logarithm, see Definition 7.38. In Chapter 7, we showed there exists a real number e, with $2 < e < 3$, such that

$$\int_1^{e^n} \frac{dx}{x} = n = \int_{e^{-n}}^1 \frac{dx}{x}.$$

In particular, the basic improper integrals

$$\int_0^1 \frac{dx}{x} \quad \text{and} \quad \int_1^\infty \frac{dx}{x}$$

both diverge.

If $p \neq 1$, the function $F(x) = x^{1-p}/(1-p)$ is an antiderivative of f on the set of positive real numbers.

(i) If $0 < p < 1$, then $1 - p > 0$, so

$$\lim_{a\to 0^+} \int_a^1 \frac{dx}{x^p} = \lim_{a\to 0^+} \frac{x^{1-p}}{1-p}\bigg|_{x=a}^{x=1} = \lim_{a\to 0^+} \frac{1-a^{1-p}}{1-p} = \frac{1}{1-p}.$$

If $p > 1$ instead, then $1 - p < 0$, and $a^{1-p} \to \infty$ as $a \to 0^+$, so the integral diverges.

(ii) If $1 < p$, then $1 - p < 0$, so

$$\lim_{b\to\infty} \int_1^b \frac{dx}{x^p} = \lim_{b\to\infty} \frac{x^{1-p}}{1-p}\bigg|_{x=1}^{x=b} = \lim_{b\to\infty} \frac{b^{1-p}-1}{1-p} = \frac{1}{p-1}.$$

If $0 < p < 1$ instead, then $1 - p > 0$, and $b^{1-p} \to \infty$ as $b \to \infty$, so the integral diverges. ☐

Definition 9.17. An improper integral $\int_a^b f$ *converges absolutely* if the improper integral $\int_a^b |f|$ converges. If the integral of f converges but the integral of $|f|$ diverges, the improper integral of f is said to *converge conditionally.*

Remark 9.18. To avoid stating results with four (or more) closely-related parts, we focus for the rest of this section on basic improper integrals over an unbounded interval $[a, \infty)$. Analogous results are true for other types of improper integral.

Proposition 9.19. *Let a be a fixed number, and assume f, g are integrable on every interval $[a, b]$ with $a < b$.*

(i) *If f is non-negative, then*

$$\int_a^\infty f \text{ converges if and only if } F(x) = \int_a^x f \text{ is bounded.}$$

(ii) *If $f^+ = \max(f, 0)$ and $f^- = -\min(f, 0)$, then*

$$\int_a^\infty f \text{ converges absolutely if and only if } \int_a^x f^\pm \text{ both converge,}$$

and this implies $\int_a^\infty f$ converges.

(iii) *If* $|g| \leq |f|$ *except at finitely many points, and if* $\int_a^\infty f$ *converges absolutely, then* $\int_a^\infty g$ *converges absolutely, and*

$$\left|\int_a^\infty g\right| \leq \int_a^\infty |f|.$$

Proof. These assertions follow immediately from the corresponding facts about ordinary (proper) integrals, infinite sequences, and series. Indeed, if (b_k) is a real sequence diverging to ∞, consider the sequence

$$B_k = \int_a^{b_k} f$$

and note that the improper integral of f converges to L if and only if every such sequence (B_k) converges to L. $\square$

Proposition 9.20. *Let* f *be improperly integrable on* $[a, \infty)$*, and let* τ *be a non-negative, differentiable function on* $[\alpha, \beta)$ *satisfying* $\tau(\alpha) = a$ *and* $\tau(x) \to \infty$ *as* $x \to \beta^-$*. Then*

$$\int_\alpha^\beta f\big(\tau(s)\big) \cdot \tau'(s)\, ds = \int_a^\infty f(t)\, dt,$$

in the sense that the left-hand integral converges and has the same value as the right-hand integral.

Proof. By the change of variables theorem for ordinary integrals,

$$\int_\alpha^x f\big(\tau(s)\big) \cdot \tau'(s)\, ds = \int_a^{\tau(x)} f(t)\, dt$$

for all $\alpha \leq x < \beta$. The claim follows by taking $x \to \beta^-$. $\square$

Remark 9.21. Generally, change of variables can be applied to improper integrals so long as the "new" variable $x = \tau(t)$ does not achieve a value near which f is unbounded.

Example 9.22. If f is continuous on $[1, \infty)$, then

$$\int_0^1 \frac{f(1/t)}{t^2}\, dt = -\int_1^0 \frac{f(1/t)}{t^2}\, dt = \int_1^\infty f(x)\, dx,$$

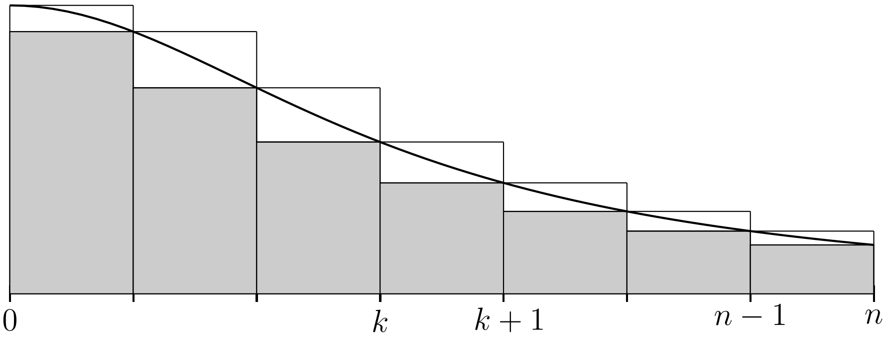
in the sense that both integrals converge or diverge, and if both converge, they have the same value.

The Integral Test for Series

Proposition 9.23. *Let f be a non-increasing, positive, real-valued function, defined for $x \geq 0$. For every positive integer n,*

$$\sum_{k=1}^{n} f(k) \leq \int_0^n f(x)\,dx \leq \sum_{k=0}^{n-1} f(k).$$

In particular, the real sequence defined by $a_k = f(k)$ for $k \geq 0$ is summable if and only if f is improperly integrable.



Proof. Let $k \geq 0$ be arbitrary. Because f is non-increasing,

$$0 < f(k+1) \leq f(x) \leq f(k) \quad \text{for } k \leq x \leq k+1.$$

Integrating over $[k, k+1]$,

$$f(k+1) \leq \int_k^{k+1} f(x)\,dx \leq f(k).$$

Summing over $k = 0, \dots, n-1$ and patching integrals,

$$\sum_{k=1}^{n} f(k) = \sum_{k=0}^{n-1} f(k+1) \leq \int_0^n f(x)\,dx \leq \sum_{k=0}^{n-1} f(k).$$

All three expressions are non-decreasing sequences in n, and the two sums differ by $f(0) - f(n)$, a finite amount, so all three converge or diverge together. □

Exercises

Exercise 9.1. The following functions are given by their value at t. Find an antiderivative, and check your answer by differentiation.
Hints: If r is non-zero, f is differentiable, and $F(t) = \frac{1}{r}f(t)^r$, then $F'(t) = f(t)^{r-1}f'(t)$. In part (e), it may help to use the identity

$$\frac{1}{1-t^2} = \frac{1}{2}\left[\frac{1}{1-t} + \frac{1}{1+t}\right].$$

(a_1) $(1+t)^3$ (a_2) $t(1+t^2)^3$ (a_3) $(1+t^2)^3$

(b_1) $\sqrt{1+t}$ (b_2) $2t\sqrt{1+t^2}$ (b_3) $t/\sqrt{2+t^2}$

(c_1) $(1+\sqrt{t})^2$ (c_2) $(t+t^{-1})^3$ (c_3) $(1-t^{-2})(t+t^{-1})^3$

(d_1) $(t+\sqrt{t})^2$ (d_2) $(\sqrt{t}+1/\sqrt{t})^2$ (d_3) $(t^{1/2}+t^{3/2})^3(1+3t)/\sqrt{t}$

(e_1) $1/(1-t^2)$ (e_2) $t/(1-t^2)$ (e_3) $1/\big((t-1)^2(1+t)\big)$

Exercise 9.2. Evaluate $\displaystyle\int_0^x \frac{(1+t)}{(2+2t+t^2)^3}\,dt$ for x in $\mathbf{R}$.

Exercise 9.3. Suppose $f : \mathbf{R} \to \mathbf{R}$ is continuous.

(a) Define $G : \mathbf{R} \to \mathbf{R}$ by

$$G(x) = \int_0^{x^2} f(t)\,dt.$$

Prove that G is differentiable, and find $G'(x)$.
Suggestion: Write G as the composition of two functions you understand well, and use the chain rule.

(b) Define $H : \mathbf{R} \to \mathbf{R}$ by

$$H(x) = \int_x^{x^2} f(t)\,dt.$$

Show that H is differentiable, and find $H'(x)$.

(c) Assume ϕ and ψ are differentiable on (α, β), and define

$$\Phi(x) = \int_{\psi(x)}^{\phi(x)} f(t)\,dt \quad \text{for } x \text{ in } (\alpha, \beta).$$

Show that Φ is differentiable, and find $\Phi'(x)$ in terms of f, ϕ, and ψ.

Exercise 9.4. Consider the functions F, $G : \mathbf{R} \to \mathbf{R}$ defined by

$$F(x) = \int_0^{x^2} \frac{t\,dt}{\sqrt[3]{1+t^3}}, \qquad G(x) = \int_0^x \frac{t^2\,dt}{\sqrt[3]{1+t^6}}.$$

(a) Calculate F' and G'.

(b) Which is larger, $F(\frac{1}{2})$ or $G(\frac{1}{2})$? Explain in detail.

Exercise 9.5. Does there exist an integrable function $f : [-1, 1] \to \mathbf{R}$ such that

$$\int_{-1}^x f(t)\,dt = \sqrt{1-x^2} \quad \text{for all } x \text{ in } [-1, 1]?$$

Explain. What if f is only required to be *improperly* integrable?

Exercise 9.6. Let $u : \mathbf{R} \to \mathbf{R}$ be continuous. Prove that for each c in $\mathbf{R}$, there is exactly one $\mathscr{C}^1$ function $f : \mathbf{R} \to \mathbf{R}$ satisfying the initial value problem

$$f' = u, \qquad f(0) = c.$$

Exercise 9.7. The natural logarithm log (Definition 7.38) is the unique function satisfying $\log' t = 1/t$ for all $t > 0$ and $\log 1 = 0$.

(a) If $r \neq 1$ is rational, use integration by parts to evaluate

$$f(x) = \int_1^x t^r \log t\,dt, \quad x > 0.$$

(b) Use substitution to evaluate the integral in part (a) if $r = -1$.

Exercise 9.8. Let $A : (-1, 1) \to \mathbf{R}$ be the $\mathscr{C}^1$ function defined by

$$A'(x) = \frac{1}{\sqrt{1-x^2}}, \qquad A(0) = 0.$$

(a) Show A is strictly increasing.

(b) By (a), A is invertible; let $S = A^{-1}$. Prove that $S' = \sqrt{1-S^2}$.

(c) Show that S is $\mathscr{C}^2$, and find S'' in terms of S.

Exercise 9.9. Use the change of variables theorem to re-prove Theorem 7.22. (Your calculation should be extremely brief. Good theoretical tools can be a tremendous labor-saving device.)

Exercise 9.10. Let $f : \mathbf{R} \to \mathbf{R}$ be continuous.

(a) Define

$$g(x) = \int_0^x tf(t)\,dt, \qquad h(x) = \int_0^x xf(t)\,dt.$$

Find $g'(x)$ and $h'(x)$.
Suggestion: Rewrite h so that there is no "x" inside the integral.

(b) Use (a) to show that

$$\int_0^x f(u)(x-u)\,du = \int_0^x \left(\int_0^u f(t)\,dt\right) du.$$

Suggestion: Differentiate each side with respect to x.

Exercise 9.11. (Uniqueness of Taylor polynomials) Let $n \geq 0$ be an integer, x_0 a real number, X an open interval containing x_0 (possibly all of $\mathbf{R}$), and f a real-valued function of class $\mathscr{C}^n$ on X.

(a) If q_n is a polynomial of degree n and $f(x) = q_n(x) + o\big((x-x_0)^n\big)$ near x_0, then q_n is the nth degree Taylor polynomial of f at x_0.

(b) If f is a polynomial of degree n, then the Taylor polynomial of f at x_0 is f itself, expanded in powers of $(x-x_0)$.

(c) Use the Taylor polynomial formula to expand a general quadratic polynomial $f(x) = ax^2 + bx + c$ in powers of $(x-x_0)$. Use algebra to verify that the result is equal to f.

(d) Expand $f(x) = x^n$ in powers of $(x-x_0)$. Show that the result is consistent with the binomial theorem.

Exercise 9.12. This exercise gives a short (but less motivated) proof of Taylor's theorem with the Lagrange remainder. Assume f is of class $\mathscr{C}^{n+1}$ on some interval X containing a point x_0. For each x in X, and each integer $n \geq 0$, define

$$\phi_{n+1}(t) = f(x) - f(t) - f'(t)(x-t) - \frac{f''(t)}{2!}(x-t)^2 - \cdots - \frac{f^{(n)}(t)}{n!}(x-t)^n.$$

Deduce Taylor's theorem by applying Rolle's theorem to the function

$$F_{n+1}(t) = \phi_{n+1}(t) - \left[\frac{x-t}{x-x_0}\right]^{n+1} \phi_{n+1}(x_0).$$

Exercise 9.13. The number z_n in the Lagrange remainder can be written uniquely as a convex linear combination $z_n = x_0 + t_n(x - x_0)$ for some real number t_n with $0 < t_n < 1$. Effectively, using t_n rescales the interval between x_0 and x to unit length.

Let f be of class $\mathscr{C}^{n+1}$ in some neighborhood of x_0, and assume $f^{(n+1)}(x_0) \neq 0$. Prove

$$t_n = \frac{1}{n+1} + o(x - x_0).$$

Hints: By the proof of Taylor's theorem, the $(n-1)$th remainder is equal to the $(n-1)$th-degree term plus the nth remainder. Rearrange, apply the Mean Value Theorem to $f^{(n)}$ on the interval between x_0 and z_n, and use the fact that $f^{(n+1)}$ has a non-zero limit at x_0.

Exercise 9.14. Recall that if f is integrable on some interval $[\alpha, \beta]$, then the *midpoint estimate* (with one subinterval) is

$$\int_\alpha^\beta f(x)\,dx \approx f(\bar{x})(\beta - \alpha), \qquad \bar{x} = \tfrac{1}{2}(\alpha + \beta),$$

the area enclosed by the horizontal line through the midpoint. This exercise bounds the error under the assumption f is of class $\mathscr{C}^2$.

(a) Use the first-order Taylor approximation with $x_0 = \bar{x}$ to prove that if $|f''(x)| \leq K_2$ for all x in $[\alpha, \beta]$, then

$$\left|\int_\alpha^\beta \big(f(x) - f(\bar{x})\big)\,dx\right| \leq \frac{K_2(\beta - \alpha)^3}{24}.$$

Hint: The integral of $f'(\bar{x})x$ over $[\alpha, \beta]$ vanishes by symmetry.

(b) Suppose f is of class $\mathscr{C}^2$ on $[a, b]$, and that $|f''(x)| \leq K_2$ for all x in $[a, b]$. Prove that if n is a positive integer, then the error for the midpoint method with n subintervals of length $\Delta x = (b - a)/n$ is at most $K_2(b - a)^3/(24n^2)$.

Exercise 9.15. Recall that if f is an integrable function on some interval $[\alpha, \beta]$, then the *trapezoid estimate* (with one subinterval) is

$$\int_\alpha^\beta f(x)\,dx \approx \tfrac{1}{2}(\beta - \alpha)\big(f(\alpha) + f(\beta)\big),$$

the area enclosed by the secant line through the endpoints. This exercise gives a bound for the error under the assumption that f is of class $\mathscr{C}^2$.

Let F be an antiderivative of f, hence of class $\mathscr{C}^3$, and introduce the functions

$$\begin{aligned} \Phi(x) &= F(x) - F(\alpha) - \tfrac{1}{2}(x-\alpha)\big(f(\alpha) + f(x)\big), \\ G(x) &= \Phi(x) - \left[\frac{x-\alpha}{\beta-\alpha}\right]^3 \Phi(\beta). \end{aligned}$$

(a) Show that $G(\alpha) = G(\beta) = 0$. Conclude there exists a z with $\alpha < z < \beta$ such that $G'(z) = 0$.

(b) Show that $G'(\alpha) = 0$. Conclude there exists a z' with $\alpha < z' < z$ such that $G''(z') = 0$, and show

$$\frac{F'''(z')}{12}(\beta-\alpha)^3 = \Phi(\beta).$$

(c) Prove that if $|f''(x)| \leq K_2$ for all x with $\alpha \leq x \leq \beta$, then

$$\left| \int_\alpha^\beta f(x)\,dx - \tfrac{1}{2}(\beta-\alpha)\big(f(\alpha)+f(\beta)\big) \right| \leq \frac{K_2}{12}(\beta-\alpha)^3.$$

(d) Suppose f is of class $\mathscr{C}^2$ on $[a, b]$, and that $|f''(x)| \leq K_2$ for all x in $[a, b]$. Prove that if n is a positive integer, then the error for the trapezoid method with n subintervals of length $\Delta x = (b-a)/n$ is at most $K_2(b-a)^3/(12n^2)$.

Exercise 9.16. Let f be a smooth, real-valued function on some open interval X, and assume there exist positive real numbers M and R such that

$$\left| \frac{f^{(n+1)}(z)R^{n+1}}{(n+1)!} \right| \leq M \quad \text{for all } z \text{ with } |z - x_0| < R.$$

Prove f is real-analytic on $B_R(x_0)$, i.e., that the remainder in Taylor's theorem converges to 0 as $n \to \infty$.

Exercise 9.17. In each part, $f(x) = \sqrt{1+x} = (1+x)^{1/2}$ for $x > -1$.

(a) Calculate the derivatives $f^{(n)}(x)$ and $f^{(n)}(0)$, and write out the Taylor series for f at the origin. (Express the derivatives in terms of factorials and powers of 2.)

(b) Show that the Lagrange form of the remainder goes to 0 in the interval $(-1, 1)$ as $n \to \infty$.

Exercise 9.18. (Newton's binomial theorem.) Let p be a rational number that is not an integer. In each part, $f(x) = (1+x)^p$ for $x > -1$.

(a) Calculate the derivatives $f^{(n)}(x)$ and $f^{(n)}(0)$, and write out the Taylor series for f at the origin. (You may leave the derivatives in "open form", as products.)

(b) Show that the Lagrange form of the remainder goes to 0 in the interval $(-1, 1)$ as $n \to \infty$.

Exercise 9.19. Suppose that the power series

$$f(x) = \sum_{k=0}^{\infty} a_k x^k = a_0 + a_1 x + a_2 x^2 + \cdots$$

is formally equal to its own derivative. Express the coefficients in terms of a_0, and find the interval of convergence of the resulting series.

Exercise 9.20. Manipulate the geometric series

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k = 1 + x + x^2 + x^3 + \cdots, \quad -1 < x < 1,$$

using differentiation and algebra, to give closed-form expressions for the power series:

(a) $\displaystyle\sum_{k=1}^{\infty} kx^k = x + 2x^2 + 3x^3 + \cdots.$

(b) $\displaystyle\sum_{k=1}^{\infty} k^2x^k = x + 4x^2 + 8x^3 + \cdots.$

Exercise 9.21. Let $n \geq 2$ be an integer, $\delta > 0$ a real number, and consider the function

$$f_n(x) = \begin{cases} \bigl(x(\delta - x)\bigr)^n & \text{if } 0 \leq x \leq \delta, \\ 0 & \text{otherwise} \end{cases}$$

(a) Prove that f_n is non-negative and of class $\mathscr{C}^{n-1}$ on $\mathbf{R}$.
Suggestion: Since $f(\delta - x) = f(x)$ for all x, and since f is defined by smooth formulas, it's enough to consider $x = 0$.

(b) Define

$$c_n = \int_0^\delta f_n(t)\,dt, \qquad F_n(x) = \frac{1}{c_n}\int_0^x f_n(t)\,dt.$$

Prove that F_n is of class $\mathscr{C}^n$, non-decreasing, and that $F_n(x) = 0$ if $x \leq 0$, $F_n(x) = 1$ if $x \geq \delta$.

(c) Assume $[c, d] \subseteq (a, b)$. Use part (b) to show there exists a non-negative function f of class $\mathscr{C}^n$ defined on $\mathbf{R}$ such that $f(x) = 1$ if $c \leq x \leq d$; $f(x) = 0$ if $x \leq a$ or $b \leq x$; and $0 \leq f(x) \leq 1$ for all real x.

Exercise 9.22. Let $f : \mathbf{R} \to \mathbf{R}$ be continuous and ℓ-periodic. Prove:

(a) For every real number a,

$$\int_a^{a+\ell} f(t)\,dt = \int_0^\ell f(t)\,dt.$$

(b) There exists an ℓ-periodic function F with $F' = f$ if and only if

$$\int_0^\ell f(t)\,dt = 0.$$

(c) The function

$$F(x) = \int_0^x f(t)\,dt - \frac{x}{\ell}\int_0^\ell f(t)\,dt$$

is ℓ-periodic.

Exercise 9.23. Determine whether or not the improper integrals converge. (Do not attempt to evaluate.)

(a) $\displaystyle\int_0^\infty \frac{dx}{1+x^2}$. (b) $\displaystyle\int_0^1 \frac{dx}{1-x^2}$. (c) $\displaystyle\int_0^\infty \frac{(x^2+1)\,dx}{x^4+1}$.

Exercise 9.24. Does the improper integral

$$\int_{-1}^1 \frac{dx}{x^2}$$

converge? What value is obtained by ignoring the fact that the integral is improper? Does this value make sense qualitatively?

Exercise 9.25. Let f be a continuous, increasing function on some interval $[a, b]$. Use the substitution $y = f(x)$ and integration by parts to prove that

$$\int_{f(a)}^{f(b)} f^{-1}(y)\,dy = bf(b) - af(a) - \int_a^b f(x)\,dx,$$

and illustrate with a sketch.

Exercise 9.26. Let F be a continuously-differentiable function on some interval I. Prove there exist non-decreasing functions F_1 and F_2 on I such that $F = F_1 - F_2$.
Suggestion: The derivative $F' = f$ is continuous, and therefore may be written as the sum of a non-negative continuous function f_1 and a non-positive continuous function f_2.

Exercise 9.27. Let F be a differentiable function on $[a, b]$, and assume the derivative $f = F'$ is integrable (though not necessarily continuous). Use the outline provided to give a proof of the Second Fundamental Theorem,

$$\int_a^b f(t)\,dt = F(b) - F(a).$$

(a) Let $P = \{t_i\}_{i=0}^{n}$ be an arbitrary partition of $[a, b]$. For each $i = 1$, 2, …, n, there exists a number t_i^* in the ith subinterval such that $F(t_i) - F(t_{i-1}) = f(t_i^*)\,\Delta t_i$.

(b) If the t_i^* are chosen as in part (a), the resulting Riemann sum is $F(b) - F(a)$.

(c) If P_n is a sequence of partitions whose maximum length goes to 0, the corresponding Riemann sums approach the integral of f over $[a, b]$.

Exercise 9.28. Let f be a positive, non-increasing function defined for $x \geq 0$, and let $a_k = f(k)$ for $k \geq 0$. Use the technique in the proof of Proposition 9.23 to find lower and upper bounds for the tail

$$\sum_{k=n+1}^{\infty} a_k = \sum_{k=n+1}^{\infty} f(k)$$

in terms of improper integrals of f.

Exercise 9.29. The sums of the infinite series

$$\sum_{k=1}^{\infty} \frac{1}{k^2}, \qquad \sum_{k=1}^{\infty} \frac{1}{k^3}, \qquad \sum_{k=1}^{\infty} \frac{1}{k^5}$$

are to be estimated to within $\varepsilon = 0.5 \times 10^{-6}$ by adding up finitely many terms. Use the preceding exercise to find a sufficient number of terms that guarantee this accuracy.

Chapter 10

The Exponential Function

Definition 7.38 introduced the natural logarithm $\log : (0, \infty) \to \mathbf{R}$:

$$\log x = \int_1^x \frac{1}{t}\,dt \quad \text{for } x > 0.$$

By Theorem 9.1, log is differentiable, with $\log' x = 1/x > 0$, and therefore is strictly increasing by Theorem 8.33. The logarithm is surjective consequent to the remarkable identities

$$\log xy = \log x + \log y, \qquad \log(1/x) = -\log x \quad \text{for all } x,\ y > 0.$$

We defined the number e, with $2 < e < 3$, by the condition $\log e = 1$.

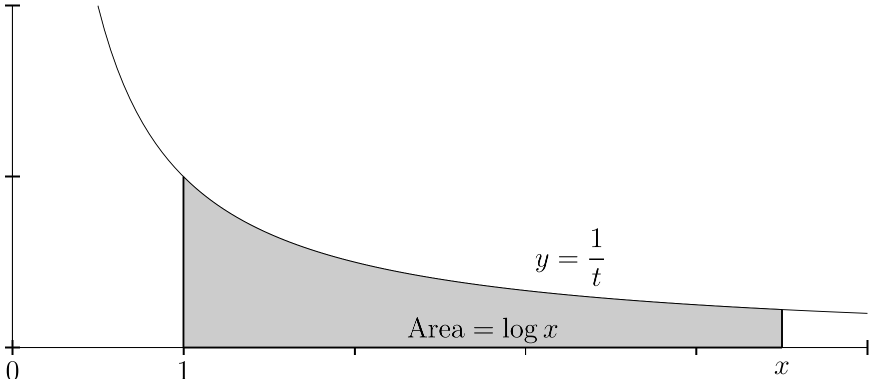


Figure 10.1: The definition of log as an integral.

Remark 10.1. The tangent lines to the graph $y = \log x$ become arbitrarily close to horizontal because $\log' x = 1/x$, but the graph has no horizontal asymptote.

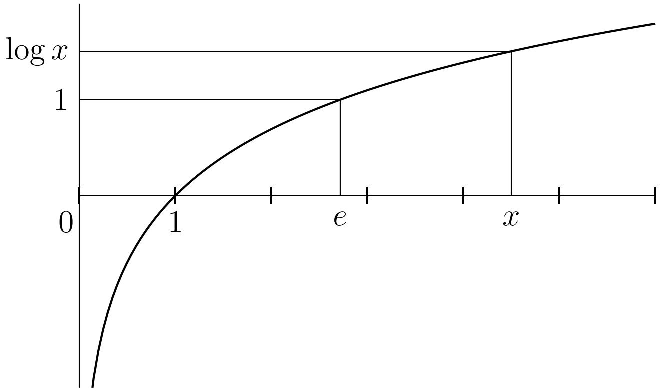


Figure 10.2: The natural logarithm function.

10.1 The Natural Exponential Function

Definition 10.2. The *natural exponential function* $\exp : \mathbf{R} \to (0, \infty)$ is the inverse of log.

Remark 10.3. That is, if y is real and $x > 0$, then $x = \exp y$ if and only if $y = \log x$. In particular, $\exp y > 0$ for all real y.

The Differential Characterization of exp

Theorem 10.4. *For all real x, $\exp' x = \exp x$. Conversely, if $f : \mathbf{R} \to \mathbf{R}$ is a differentiable function such that $f' = f$, then $f(x) = f(0) \exp x$ for all x.*

Proof. For all real x, we have $x = \log(\exp x)$. Since log is differentiable and has non-vanishing derivative, its inverse function exp is differentiable, and by Theorem 8.27,

$$\exp' x = \frac{1}{\log'(\exp x)} = \frac{1}{1/\exp x} = \exp x.$$

Conversely, suppose $f : \mathbf{R} \to \mathbf{R}$ is differentiable and $f' = f$. Since $\exp x > 0$ for all real x, we may consider $g(x) = f(x)/\exp x$, a quotient of differentiable functions. Since $\exp' = \exp$, the quotient rule gives

$$g'(x) = \frac{(\exp x)f'(x) - f(x)(\exp x)}{(\exp x)^2} = \frac{f'(x) - f(x)}{\exp x} = 0 \quad \text{for all } x.$$

By the identity theorem, g is a constant function: $g(x) = g(0) = f(0)$ for all real x. That is, $f(x) = f(0) \exp x$ for all x. □

Real Exponentiation

Proposition 10.5. *If $a > 0$ is real and $r = p/q$ is rational, then*

$$\log a^r = r \log a, \quad \textit{i.e.,} \quad a^r = \exp(r \log a).$$

Proof. Mathematical induction together with $\log xy = \log x + \log y$ shows that $\log b^m = m \log b$ for every positive real number b and every integer m.

Set $b = a^r = a^{p/q}$, so that $b^q = a^p$. Taking logarithms and using the preceding paragraph, $q \log b = p \log a$, or

$$\log a^r = \log b = (p/q) \log a = r \log a. \qquad \square$$

Proposition 10.5 suggests the definition of exponentiation with arbitrary real exponent.

Definition 10.6. If $a > 0$ and x is real, we define

$$a^x = \exp(x \log a).$$

In particular, $e^x = \exp x$ for all real x.

Theorem 10.7. *Let r be a real number, and define $f(x) = x^r$ for $x > 0$. The function f is differentiable, and $f'(x) = rx^{r-1}$.*

Proof. By definition, $f(x) = \exp(r \log x)$. The chain rule gives

$$f'(x) = \exp'(r \log x)\, r \log' x = x^r (r/x) = rx^{r-1}. \qquad \square$$

Theorem 10.8. *Let $b > 0$. The function $\exp_b x = b^x = \exp(x \log b)$ is differentiable, and*

$$\exp_b' x = (\log b) \exp_b x \quad \textit{for all real } x.$$

Conversely, if k is a real number and f is a differentiable function satisfying $f' = kf$, then $f(x) = f(0) \exp(kx)$ for all x.

Proof. If k is real and $f(x) = \exp(kx)$, Theorem 10.4 and the chain rule give $f'(x) = kf(x)$. The first assertion follows by taking $k = \log b$.

The converse is proven analogously to Theorem 10.4, and is left to you, see Exercise 10.21. $\square$

Corollary 10.9. *$e^{x+y} = e^x e^y$ and $e^{xy} = (e^y)^x$ for all real x, y.*

Proof. Fix a real number y arbitrarily, and define $f(x) = e^{x+y}$. By the chain rule, f is differentiable, and $f' = f$. Since $f(0) = e^y$, Theorem 10.4 implies $e^{x+y} = e^x e^y$ for all real x.

To prove the second assertion, consider $g(x) = (e^y)^x$. By the first part of Theorem 10.8 with $b = e^y$,

$$g'(x) = \log(e^y)(e^y)^x = yg(x).$$

By the second part of Theorem 10.8 $g(x) = e^{xy}$, since $g(0) = 1$.

As a fringe benefit, $(e^y)^x = (e^x)^y$, since each is equal to e^{xy}. ☐

Remark 10.10. The identity $\exp(x+y) = \exp x \exp y$ may also be proven algebraically. Set $u = \exp x$ and $v = \exp y$. Applying exp to both sides of the identity $x + y = \log u + \log v = \log uv$ gives

$$\exp(x + y) = \exp(\log uv) = uv = \exp x \exp y.$$

Corollary 10.11. *If $b > 0$ is real, then $b^{x+y} = b^x b^y$ and $b^{xy} = (b^x)^y$ for all real x, y.*

The proof is requested in Exercise 10.22.

Remark 10.12. We do not define $0^0 = 1$ without qualification. Indeed, for $0 \leq L \leq 1$ we can "make $0^0 = L$". If $\alpha < 0$, then

$$\lim_{x\to 0} \exp(\alpha/x^2)^{x^2} = \lim_{x\to 0} \big(e^{(\alpha/x^2)}\big)^{x^2} = e^\alpha < 1,$$

and similarly, $\lim\limits_{x\to 0} \exp(-1/x^4)^{x^2} = 0$.

Definition 10.13. Let $b > 0$ be real. The inverse of $\exp_b$, denoted $\log_b$, is the *logarithm to the base b*.

Proposition 10.14. *Fix $b > 0$. We have $\log_b x = (\log x)/(\log b)$ for all positive x.*

Proof. The equation $y = \log_b x$ means $x = b^y = \exp(y \log b)$. Taking the natural logarithm, $y = (\log x)/(\log b)$. ☐

10.2 Representations of exp

Theorem 10.15. *For all real x,*

$$e^x = \lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n.$$

Proof. The theorem is obvious if $x = 0$. Otherwise, if $0 < h < 1/|x|$ then $1 + xh > 0$. Since $\log 1 = 0$, we may write

$$\frac{1}{h}\log(1+xh) = \frac{\log(1+xh) - \log 1}{h} = x\frac{\log(1+xh) - \log 1}{xh}.$$

As $h \to 0$, this approaches $x \log' 1 = x$. Writing $h = 1/n$ and using continuity of exp,

$$\begin{aligned}\lim_{n\to\infty}\left(1 + \frac{x}{n}\right)^n &= \lim_{h\to 0^+}(1+xh)^{1/h}\\ &= \lim_{h\to 0^+}\exp\left[\frac{1}{h}\log(1+xh)\right]\\ &= \exp\left[\lim_{h\to 0^+}\frac{1}{h}\log(1+xh)\right] = e^x.\end{aligned}$$

□

Remark 10.16. Theorem 10.15 characterizes the natural exponential function as a limit of geometric growth. If, for example, x is the annual interest rate on a savings account, and there are n compoundings per year, then the multiplier on the right gives the factor by which the savings increase over one year. As the number of compoundings per year grows without bound, the balance does not become infinite in a finite time. Instead, if \$1 is allowed to accrue interest with continuous compounding, then in the time it would take the savings to double without compounding, the balance increases to \$2.72 (rounded to the nearest penny).

Theorem 10.17. $e^x = \displaystyle\sum_{k=0}^{\infty}\frac{x^k}{k!}$ *for all real* x.

Proof. Let $f(x)$ denote the sum of the power series. The coefficients are $a_k = 1/k!$, and

$$\lim_{k\to\infty}\left|\frac{a_{k+1}x^{k+1}}{a_k x^k}\right| = \lim_{k\to\infty}\frac{k!\,|x|}{(k+1)!} = \lim_{k\to\infty}\frac{|x|}{k+1} = 0.$$

Since this is less than 1 independently of x, the series converges absolutely for all x. By Theorem 9.10, the series may be differentiated term by term:

$$f'(x) = \sum_{k=1}^{\infty}\frac{1}{k!}\,kx^{k-1} = \sum_{k=0}^{\infty}\frac{x^k}{k!} = f(x).$$

Since $f(0) = 1$, Theorem 10.4 implies $f(x) = e^x$ for all x. □

Remark 10.18. The exponential series could have been found using Taylor's theorem since $f^{(n)}(0) = \exp 0 = 1$ for all n. The series can be easily remembered: The constant term is $1 = e^0$, and each successive term is the integral from 0 of the preceding term.

Corollary 10.19. $e = \displaystyle\sum_{k=0}^{\infty} \frac{1}{k!} = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \dots$ *is irrational.*

Proof. The series representation follows by setting $x = 1$ in the exponential power series. We will show that if e is rational, then there exist integers N and M such that $N < M < N + 1$. Contrapositively, since such integers obviously do not exist, e is irrational.

If ℓ and m are positive integers, Lemma 2.30 gives

$$\frac{1}{(\ell + m)!} \leq \frac{1}{(\ell + 1)^m\, \ell!},$$

with strict inequality if $m > 1$. By the geometric series formula,

$$\sum_{m=1}^{\infty} \frac{1}{(\ell + 1)^m} = \frac{1/(\ell + 1)}{1 - 1/(\ell + 1)} = \frac{1}{(\ell + 1) - 1} = \frac{1}{\ell},$$

and so

$$\sum_{k=\ell+1}^{\infty} \frac{1}{k!} = \sum_{m=1}^{\infty} \frac{1}{(\ell + m)!} < \sum_{m=1}^{\infty} \frac{1}{(\ell + 1)^m\, \ell!} = \frac{1}{\ell \cdot \ell!}.$$

Since every term in the series for e is positive, we have

$$\sum_{k=0}^{\ell} \frac{1}{k!} < e = \sum_{k=0}^{\infty} \frac{1}{k!} = \sum_{k=0}^{\ell} \frac{1}{k!} + \sum_{k=\ell+1}^{\infty} \frac{1}{k!} < \sum_{k=0}^{\ell} \frac{1}{k!} + \frac{1}{\ell \cdot \ell!}$$

for every positive integer ℓ. Multiplying through by $\ell!$,

$$\sum_{k=0}^{\ell} \frac{\ell!}{k!} < \ell!\, e < \sum_{k=0}^{\ell} \frac{\ell!}{k!} + \frac{1}{\ell}.$$

Each term $\ell!/k!$ of the sum is an integer since $0 \leq k \leq \ell$, so the sum represents some integer $N(\ell)$ depending on ℓ.

If e is rational, then there exist positive integers p and q such that $e = p/q$. But then $N(q) = N$ and $q!\, e = p(q - 1)! = M$ are integers, and the preceding inequality implies $N < M < N + (1/q) < N + 1$. Clearly, no such integers M and N exist. $\square$

Remark 10.20. The preceding proof shows that for every integer $\ell \geq 1$,

$$e = \left(\sum_{k=0}^{\ell} \frac{1}{k!}\right) + \frac{1}{2\ell \cdot \ell!} + \mathsf{A}\left(\frac{1}{2\ell \cdot \ell!}\right).$$

For example, taking $\ell = 10$ gives a rational estimate for e that is accurate to within $\varepsilon = 1/(20 \cdot 10!) = 1/72,576,000 < 1.4 \times 10^{-8}$:

$$e \approx \frac{197,282,021}{72,576,000} + \mathsf{A}(\varepsilon) \approx 2.718281815 + \mathsf{A}(\varepsilon).$$

(To fifteen decimals, the value of e is $2.71828\,18284\,59045\dots$.)

10.3 Hyperbolic Functions

Definition 10.21. The *hyperbolic cosine* and *hyperbolic sine* functions are defined to be the even and odd parts of exp, respectively:

$$\cosh x = \frac{e^x + e^{-x}}{2}, \qquad \sinh x = \frac{e^x - e^{-x}}{2}.$$

The *hyperbolic tangent* and *hyperbolic secant* functions are defined by

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}, \qquad \operatorname{sech} x = \frac{1}{\cosh x}.$$

Remark 10.22. The functions "cosh" and "sech" are pronounced phonetically; sinh is pronounced "cinch" or "shine", and tanh rhymes with "ranch". As the notation suggests, these functions have a close kinship with the circular functions of ordinary trigonometry.

Proposition 10.23. *The functions* cosh, sinh, tanh, *and* sech *are defined for all real numbers, and satisfy the identities* $\cosh^2 - \sinh^2 = 1$,

$$\cosh' = \sinh, \qquad \sinh' = \cosh, \qquad \tanh' = \operatorname{sech}^2.$$

cosh *maps* $[0, \infty)$ *bijectively to* $[1, \infty)$. sinh *maps* $\mathbf{R}$ *bijectively to* $\mathbf{R}$. tanh *maps* $\mathbf{R}$ *bijectively to* $(-1, 1)$.

Every point (x, y) *on the right nappe of the hyperbola* $x^2 - y^2 = 1$ *is of the form* $(\cosh t, \sinh t)$ *for a unique real* t.

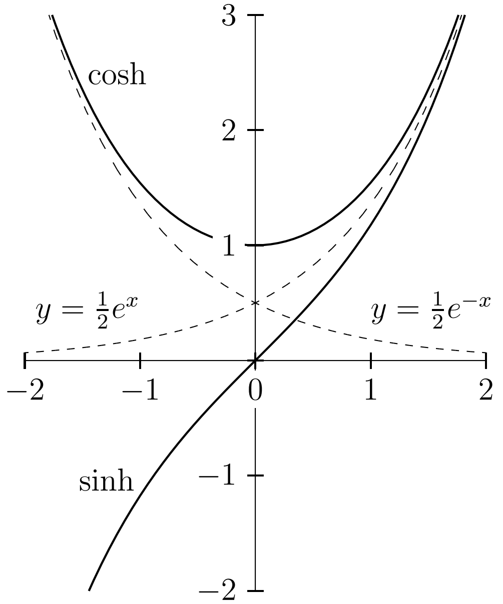


Figure 10.3: Hyperbolic cosine and sine.

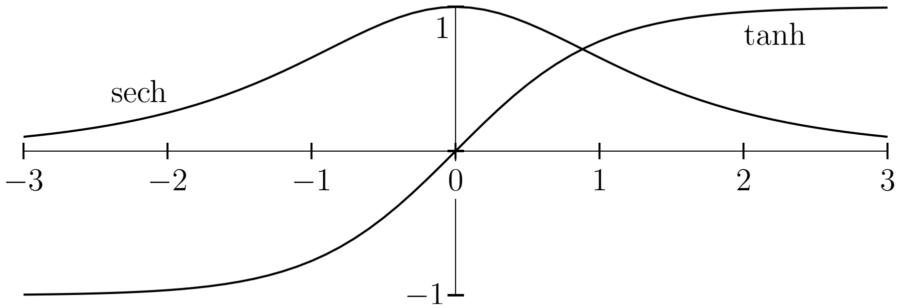


Figure 10.4: Hyperbolic tangent and secant.

Proposition 6.47 and the power series for exp immediately give series expansions for cosh and sinh.

Proposition 10.24. *For all real x,*

$$\begin{aligned}\cosh x &= \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!} &&= 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \dots,\\ \sinh x &= \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!} &&= x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \dots.\end{aligned}$$

Proposition 10.25. *For all real x and y,*

$$\begin{aligned}\cosh(x+y) &= \cosh x\cosh y + \sinh x\sinh y,\\ \sinh(x+y) &= \cosh x\sinh y + \sinh x\cosh y,\\ \tanh(x+y) &= \frac{\tanh x + \tanh y}{1+\tanh x\tanh y}.\end{aligned}$$

Inverse Hyperbolic Functions

Proposition 10.26. *The inverse hyperbolic functions are as follows:*

$$\begin{aligned}\cosh^{-1} x &= \log\bigl(x \pm \sqrt{x^2-1}\bigr), &&x\geq 1;\\ \sinh^{-1} x &= \log\bigl(x + \sqrt{x^2+1}\bigr), &&x \text{ real;}\\ \tanh^{-1} x &= \frac{1}{2}\log\left(\frac{1+x}{1-x}\right), &&-1<x<1.\end{aligned}$$

Their derivatives are

$$\begin{aligned}(\cosh^{-1})'(x) &= \frac{\pm 1}{\sqrt{x^2-1}}, &&x\geq 1;\\ (\sinh^{-1})'(x) &= \frac{1}{\sqrt{x^2+1}}, &&x \text{ real;}\\ (\tanh^{-1})'(x) &= \frac{1}{1-x^2}, &&-1<x<1.\end{aligned}$$

Remark 10.27. Because cosh is even, the two branches of $\cosh^{-1}$ must be negatives of each other. By the difference of squares identity,

$$x-\sqrt{x^2-1} = \frac{1}{x+\sqrt{x^2-1}} > 0 \quad \text{for } x\geq 1,$$

so indeed, $\log\bigl(x\pm\sqrt{x^2-1}\bigr) = \pm\log\bigl(x+\sqrt{x^2-1}\bigr)$. Similar arguments show $\sinh^{-1}$ and $\tanh^{-1}$ are odd.

Proof. If $x\geq 1$, then $x=\cosh y = (e^y+e^{-y})/2$ if and only if

$$(e^y)^2-(2x)e^y+1=0.$$

This is a quadratic in e^y. By the quadratic formula, $e^y = x\pm\sqrt{x^2-1}$, or

$$y=\cosh^{-1} x = \log\bigl(x\pm\sqrt{x^2-1}\bigr).$$

Similarly, $x = \sinh y = (e^y - e^{-y})/2$ if and only if

$$(e^y)^2 - (2x)e^y - 1 = 0.$$

The quadratic formula gives $e^y = x + \sqrt{x^2+1}$, or

$$y = \sinh^{-1} x = \log\bigl(x + \sqrt{x^2+1}\bigr).$$

There is no sign ambiguity because only this choice leads to a real-valued function when x is real.

Finally,

$$x = \tanh y = \frac{e^y - e^{-y}}{e^y + e^{-y}} = \frac{e^{2y}-1}{e^{2y}+1}$$

if and only if $xe^{2y} + x = e^{2y} - 1$, or $e^{2y} = (1+x)/(1-x)$, or

$$y = \tanh^{-1} x = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right).$$

The derivatives are found by direct calculation. □

10.4 Factorials

An Exponential Estimate

Factorials are closely connected with the natural exponential function. The following is a weak version of *Stirling's formula.*

Proposition 10.28. *If $n \geq 1$ is an integer, then*

$$e^{7/8}\left(\frac{n}{e}\right)^n \sqrt{n} < n! < e\left(\frac{n}{e}\right)^n \sqrt{n}.$$

Proof. Integrating by parts with $u = \log x$, we have

$$\int_1^n \log x\,dx = x\log x - x\bigg|_{x=1}^{n} = n\log n - n + 1 = \log\left(\frac{n^n}{e^n}\cdot e\right).$$

The natural logarithm function is concave, so its graph lies above every secant line and below every tangent line.

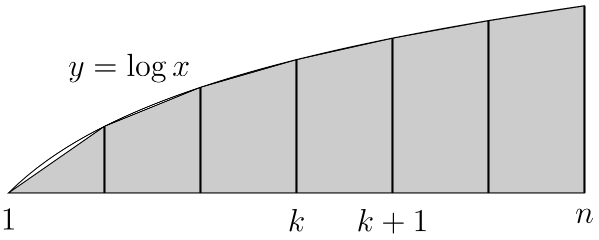


Figure 10.5: Secant line approximation of the integral.

Subdivide $[1, n]$ into $n - 1$ subintervals of length 1. The total area enclosed by the resulting secant lines is smaller than the integral of log, so we have

$$\begin{aligned}\tfrac{1}{2}(\log 1 + \log 2) + \tfrac{1}{2}(\log 2 + \log 3) + \cdots + \tfrac{1}{2}\bigl(\log(n-1) + \log n\bigr) \\ = -\tfrac{1}{2}\log n + \sum_{k=1}^{n} \log k = \log\left(\frac{n!}{\sqrt{n}}\right) < \log\left(\frac{n^n}{e^n} \cdot e\right).\end{aligned}$$

Exponentiating and rearranging gives the asserted upper bound for $n!$.

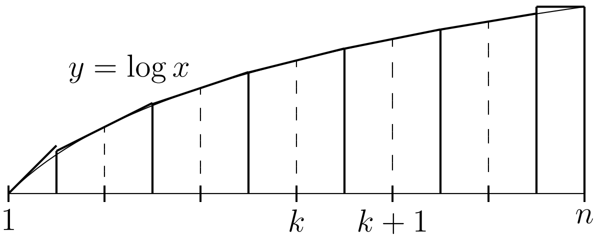


Figure 10.6: Tangent line approximation of the integral.

For each integer $k = 2, 3, \dots, n-1$, construct the trapezoid of width 1 whose top edge is the tangent line to the graph $y = \log x$ at k. The area of this trapezoid is $\log k$.

The remaining pieces of integral are bounded above by

$$\int_1^{1/2} \log x\,dx < \int_1^{1/2} (x-1)\,dx = 1/8, \qquad \int_{n-(1/2)}^{n} \log x\,dx < (\log n)/2.$$

The total area of these regions is

$$\tfrac{1}{8} + \tfrac{1}{2}\log n + \sum_{k=1}^{n-1} \log k = \tfrac{1}{8} - \tfrac{1}{2}\log n + \sum_{k=1}^{n} \log k = \log\left(e^{1/8} \cdot \frac{n!}{\sqrt{n}}\right).$$

Consequently,

$$\log\left(\frac{n^n}{e^n}\cdot e\right) < \log\left(e^{1/8}\cdot\frac{n!}{\sqrt{n}}\right),$$

and rearranging gives the asserted lower bound for $n!$. □

The Gamma Function

The Gamma function is a "smooth interpolation" of factorials for positive real numbers. The treatment here follows an exercise in Rudin.

Lemma 10.29. *For every real number $x > 0$, $t^x/e^t \to 0$ as $t \to \infty$.*

Proof. The series representation for e^t guarantees that if $n \geq x > 0$ is an integer and $t > 0$ is real, then $e^t > t^{n+1}/(n+1)!$. Consequently, if $t > 0$, then $t^x/e^t \leq t^n/e^t < (n+1)!/t$, and the upper bound goes to 0 as $t \to \infty$. □

Proposition 10.30. *For every real $x > 0$, the improper integral*

$$\int_0^\infty t^{x-1}e^{-t}\,dt$$

converges.

Proof. Since $e^{-t} < 1$ for $t > 0$,

$$\int_0^1 t^{x-1}e^{-t}\,dt < \int_0^1 t^{x-1}\,dt = \frac{1}{x},$$

so the improper integral on the left converges.

If $n \geq x-1$ is a positive integer, then $e^t > t^{n+2}/(n+2)!$ for $t > 0$ by the same idea as in the proof of Lemma 10.29. Thus

$$\int_1^\infty t^{x-1}e^{-t}\,dt \leq \int_1^\infty t^n e^{-t}\,dt \leq (n+2)!\int_1^\infty t^{-2}\,dt = (n+2)!,$$

so the improper integral on the left converges. □

Definition 10.31. The *Gamma function* $\Gamma : (0,\infty) \to \mathbf{R}$ is defined by

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,dt \quad \text{for } x > 0.$$

The main result of this section is a characterization, due to Bohr and Mollerup, of Γ in simple abstract terms.

Theorem 10.32. *The function* Γ *satisfies the following properties:*

(i) $\Gamma(x+1) = x\Gamma(x)$ *for all* $x > 0$.

(ii) $\Gamma(n+1) = n!$ *for all integers* $n > 0$.

(iii) $\log\Gamma$ *is convex.*

Conversely, if $f : (0,\infty) \to \mathbf{R}$ *satisfies these properties, then* $f = \Gamma$.

Proof. (i) Fix $b > 0$. Integrating by parts with $u = t^x$ and $dv = e^{-t}\,dt$,

$$\begin{aligned}\int_0^b t^x e^{-t}\,dt &= -t^x e^{-t}\Big|_{t=0}^{t=b} + x\int_0^b t^{x-1}e^{-t}\,dt\\ &= -b^x e^{-b} + x\int_0^b t^{x-1}e^{-t}\,dt.\end{aligned}$$

By Lemma 10.29, $-b^x e^{-b} \to 0$ as $b \to \infty$, so

$$\Gamma(x+1) = \int_0^\infty t^x e^{-t}\,dt = x\int_0^\infty t^{x-1}e^{-t}\,dt = x\Gamma(x).$$

(ii) If $n = 0$, we have

$$\Gamma(n+1) = \int_0^\infty e^{-t}\,dt = \lim_{b\to\infty} -e^{-t}\Big|_{t=0}^{t=b} = \lim_{b\to\infty}(1 - e^{-b}) = 1 = 0!.$$

By part (a) and mathematical induction, $\Gamma(n+1) = n!$ for $n \geq 0$.

(iii) If $\frac{1}{p} + \frac{1}{q} = 1$, then the integrand of $\Gamma\big(\frac{x}{p} + \frac{y}{q}\big)$ may be written

$$t^{\frac{x}{p}+\frac{y}{q}-1}e^{-t} = t^{\frac{x}{p}+\frac{y}{q}-\frac{1}{p}-\frac{1}{q}}e^{-t(\frac{1}{p}+\frac{1}{q})} = \big(t^{x-1}e^{-t}\big)^{1/p}\big(t^{y-1}e^{-t}\big)^{1/q}.$$

Since $t^{x-1}e^{-t} > 0$ for $t > 0$, Hölder's inequality (Exercise 10.27) implies

$$\Gamma\big(\tfrac{x}{p} + \tfrac{y}{q}\big) \leq \left(\int_0^\infty t^{x-1}e^{-t}\,dt\right)^{1/p}\left(\int_0^\infty t^{y-1}e^{-t}\,dt\right)^{1/q} = \Gamma(x)^{1/p}\Gamma(y)^{1/q}.$$

Taking logarithms,

$$\log\Gamma\big(\tfrac{x}{p} + \tfrac{y}{q}\big) \leq \tfrac{1}{p}\log\Gamma(x) + \tfrac{1}{q}\log\Gamma(y),$$

which is the desired convexity statement.

Conversely, suppose $f : (0,\infty) \to \mathbf{R}$ satisfies (a), (b), and (c), and set $\varphi = \log f$. Property (a) says $\varphi(x+1) = \varphi(x) + \log x$ for $x > 0$; that is, the difference quotient of φ on $[y, y+1]$ is $\log y$. By induction on n,

$$(*) \qquad \varphi(x+n+1) = \varphi(x) + \log\bigl((n+x)(n-1+x)\cdots(1+x)x\bigr).$$

Property (b) says $\varphi(n+1) = \log(n!)$ for every positive integer n and (c) says φ is convex.

Fix a positive integer n. For $0 < x < 1$, consider the four points $n < n+1 < n+1+x < n+2$. Since φ is convex, the difference quotients of φ on the intervals $[n, n+1]$, $[n+1, n+1+x]$, and $[n+1, n+2]$ are in non-decreasing order, i.e.,

$$\log n \le \frac{\varphi(n+1+x) - \varphi(n+1)}{x} \le \log(n+1).$$

Multiplying by x and substituting (*),

$$x\log n \le \varphi(x) + \log\bigl((n+x)(n-1+x)\cdots x\bigr) - \log(n!) \le x\log(n+1),$$

or

$$0 \le \varphi(x) - \log\frac{n!\,n^x}{x(x+1)\dots(x+n-1)(x+n)} \le x\log\left(1+\frac{1}{n}\right).$$

As $n \to \infty$, the upper bound goes to 0, so

$$\varphi(x) = \lim_{n\to\infty} \log\frac{n!\,n^x}{x(x+1)\dots(x+n-1)(x+n)}$$

for $0 < x < 1$. By property (a), φ satisfies this equation for all $x > 0$.

We have shown that if f satisfies properties (a)–(c), then $\log f$ is the limit on the right. Since Γ satisfies (a)–(c), the limit on the right must be equal to $\log\Gamma(x)$. □

Remark 10.33. In particular, we obtain the equation

$$\Gamma(x) = \lim_{n\to\infty} \frac{n!\,n^x}{x(x+1)\dots(x+n-1)(x+n)} \quad \text{for } x > 0,$$

analogous to the characterization of exp in Theorem 10.15.

Corollary 10.34. *If $a > 0$, then* $\displaystyle\int_0^\infty t^{x-1}e^{-t/a}\,dt = a^x \cdot \Gamma(x)$.

Proof. Making the substitution $t = au$,

$$\int_0^\infty t^{x-1}e^{-t/a}\,dt = a^{x-1}\cdot a\int_0^\infty u^{x-1}e^{-u}\,du = a^x \cdot \Gamma(x).$$ □

Exercises

Exercise 10.1. Let a, b, and c be positive real numbers. Is it more sensible to agree that a^{b^c} is equal to $(a^b)^c$ or to $a^{(b^c)}$, or does it matter?

Exercise 10.2. Prove that $x^{\log y} = y^{\log x}$ for all x, $y > 0$.

Exercise 10.3. (a) Let n and N be positive integers. Prove that

$$n \leq \log_{10} N < n + 1 \quad \text{if and only if} \quad 10^n \leq N < 10^{n+1},$$

if and only if N is an integer having $n+1$ digits. In words, the integer part of the base 10 logarithm of N is one less than the number of digits of N.

(b) Which is larger, $2^{2^{2^{2^{2^2}}}}$ or $10^{10^{100}}$? About how many digits does each number have?

Exercise 10.4. Fix $b > 1$, and define a sequence $(x_k)_{k=0}^{\infty}$ by

$$x_0 = 1, \quad x_{k+1} = b^{x_k} \quad \text{for } k \geq 0.$$

Thus $x_1 = b$, $x_2 = b^b$, $x_3 = b^{b^b}$, and so forth.

(a) Prove that if $(x_k) \to \ell$, then $b^\ell = \ell$.

(b) Use part (a) to determine the set of b for which (x_k) converges.

(c) Show ℓ increases with b, and find the largest possible value of ℓ. Suggestion: Express b as a function of ℓ.

(d) Two people are arguing. One says that if $b = \sqrt{2}$, then $\ell = 2$, since $\sqrt{2}^2 = 2$; the other says $\ell = 4$ because $\sqrt{2}^4 = 4$. Who (if either) is correct, and why?

Exercise 10.5. Let u be differentiable. Find the derivative of $\exp \circ u$, and the derivative of $\log \circ |u|$ at points where u is non-zero.

Exercise 10.6. Let n be a positive integer, and define $f(x) = x^n e^{-x}$ for $x \geq 0$. Prove f has a maximum value, and find it with justification.

Exercise 10.7. Let $f(x) = e^{-x}(x^2 - 1)$ for x real.

(a) Sketch the graph of f, using information about the first two derivatives to determine intervals on which f is monotone, convex, and concave.
Suggestion: Introduce symbolic constants to simplify calculations.

(b) Let y be real. Determine, with justification, how many solutions does the equation $f(x) = y$ has. (The answer depends on y.)

Exercise 10.8. Finish the proof of Proposition 10.26 by calculating the derivatives of $\cosh^{-1}$, $\sinh^{-1}$, and $\tanh^{-1}$.

Exercise 10.9. Establish the given identities:
(a) $\cosh(\sinh^{-1} t) = \sqrt{t^2+1}$. (b) $\sinh(\cosh^{-1} t) = \sqrt{t^2-1}$, $|t| \geq 1$.

Exercise 10.10. Use the substitution $u = \sinh t$ to show

$$\int_0^x \sqrt{u^2+1}\,du = \tfrac{1}{2}\big(x\sqrt{x^2+1} + \log(x + \sqrt{x^2+1})\big).$$

Exercise 10.11. Evaluate $\displaystyle\int_1^{\cosh t} \sqrt{u^2-1}\,du$.

Exercise 10.12. Evaluate $\displaystyle\int_{-\infty}^{\infty} \operatorname{sech}^2 x\,dx$.

Exercise 10.13. For $x > -1$ real, define

$$F(x) = \int_0^x \frac{e^{t^2}}{\sqrt{t^2+1}}\,dt, \qquad G(x) = \int_0^{x^2} \frac{e^t}{\sqrt{t+1}}\,dt.$$

Which is larger, $F(1/2)$ or $G(1/2)$?

Exercise 10.14. Let $n \geq 0$ be an integer. Show that

$$\int_0^1 (\log x)^n\,dx = (-1)^n n!.$$

Exercise 10.15. Prove that $\displaystyle\int_0^{\infty} e^{-t^2}\,dt$ converges. (Do not evaluate.)

Exercise 10.16. Granted that $\displaystyle\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$, evaluate:

(a) $\displaystyle\int_0^{\infty} \frac{e^{-x}}{\sqrt{x}}\,dx = \Gamma(1/2)$. (b) $\displaystyle\int_{-\infty}^{\infty} e^{-x^2/a^2}\,dx \quad (a > 0)$.

(c) $\displaystyle\int_0^{\infty} x^n e^{-x^2/a^2}\,dx$ and $\displaystyle\int_{-\infty}^{\infty} x^n e^{-x^2/a^2}\,dx \quad (a > 0)$.

Exercise 10.17. Use appropriate substitutions and algebra to evaluate:
(a) $\lim\limits_{x\to 0^+} x\log x$. (b) $\lim\limits_{x\to 0^+} x^x$. (c) $\lim\limits_{x\to\infty} x^{1/x}$. (d) $\lim\limits_{x\to\infty}(1+x)^{1/x}$.

Exercise 10.18. Define $f(x) = x^x$ for $x > 0$.

(a) Calculate f', and find $\lim(f', 0^+)$.

(b) Prove that f has a unique minimum, and find the location and value of the minimum.

Exercise 10.19. Let $n > 0$ be an integer. Prove that $\lim\limits_{x\to\infty} \dfrac{(\log x)^n}{x} = 0$.

Exercise 10.20. Determine (with proof, of course) which of the following converge; do not attempt to evaluate the sums!
(a) $\displaystyle\sum_{n=1}^{\infty} \frac{\log n}{n^{3/2}}$, (b) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{\log n}{n}$, (c) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\log n)^2}$,
(d) $\displaystyle\sum_{n=1}^{\infty} \frac{(n+1)^n}{n^{n+1}}$, (e) $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n}$.

Hint for (a): Use the preceding exercise, and "borrow" a small power of n to nullify the log.

Exercise 10.21. Complete the proof of Theorem 10.8: If k is a real number and f is a differentiable function satisfying $f' = kf$, then $f(x) = f(0)\exp(kx)$ for all x.

Exercise 10.22. Prove Corollary 10.11: If $b > 0$ is real, then $b^{x+y} = b^x b^y$ and $b^{xy} = (b^x)^y$ for all real x, y.

Exercise 10.23. Use the exponential estimate to find lower and upper bounds for $(2n)!/(n!)^2$ that do not contain factorials.

Exercise 10.24. Using the estimate of Remark 10.20, what is the smallest number of terms guaranteeing an accuracy $\varepsilon < 0.5 \times 10^{-1000}$?

Exercise 10.25. Define $f : \mathbf{R} \to \mathbf{R}$ by

$$f(x) = \begin{cases} \exp(-1/x^2) & \text{if } x > 0 \\ 0 & \text{if } x \le 0 \end{cases}$$

Prove that $f^{(k)}(0)$ exists and is equal to zero for all $k \ge 0$.
Suggestion: First use induction on the degree to prove that for every polynomial p,

$$\lim_{x\to 0} p(1/x)\, f(x) = 0.$$

Then show inductively that every derivative of f is of this form.

Exercise 10.26. This question asks you to mimic the estimates in the proof of Proposition 10.28 for the function $f(x) = 1/x$ on the interval $[1, n]$, with $n \geq 1$ an integer.

(a) Use secant lines on the intervals $[k, k+1]$ to show that

$$\log n < \frac{1}{2} + \left(\sum_{k=2}^{n-1} \frac{1}{k}\right) + \frac{1}{2n}.$$

(b) Use tangent lines at $k = 2, 3, \dots, n-1$, and suitably handle the intervals at the ends, to show that

$$\log 3/2 + \left(\sum_{k=2}^{n-1} \frac{1}{k}\right) + \frac{1}{2n} < \log n.$$

(c) Prove that $\displaystyle\left(\sum_{k=1}^{n} \frac{1}{k}\right) - \log n$ is increasing and bounded above. (The limit is called the *Euler-Mascheroni constant*, γ.)

Exercise 10.27. This exercise outlines a proof of *Hölder's inequality*: If p, $q > 1$ are real numbers such that $\frac{1}{p} + \frac{1}{q} = 1$, and if f and g are integrable on $[a, b]$, then

$$\left|\int_a^b fg\right| \leq \int_a^b |fg| \leq \left(\int_a^b |f|\right)^{1/p} \left(\int_a^b |g|\right)^{1/q}.$$

(a) Let $0 < \alpha < 1$. Prove that $t^\alpha \leq \alpha t + (1 - \alpha)$ for all $t \geq 0$.

(b) Let $\beta = 1 - \alpha$, so that $0 < \beta < 1$. Prove that

$$u^\alpha v^\beta \leq \alpha u + \beta v \quad \text{for all } u,\ v > 0.$$

Suggestion: Set $t = u/v$ in part (a).

(c) Let $p > 1$, and set $q = p/(p-1)$, so that $\frac{1}{p} + \frac{1}{q} = 1$. Show that

$$AB \leq \tfrac{1}{p}A^p + \tfrac{1}{q}B^q \quad \text{for all } A,\ B \geq 0.$$

Suggestion: Use part (b) with appropriate changes of variable.

(d) Prove Hölder's inequality. Suggestion: Use part (c) with A and B suitable integrals.

Exercise 10.28. This exercise outlines a second proof that e is irrational. The argument is more involved than the proof in the text, but generalizes easily to show e is transcendental (not the root of a polynomial with integer coefficients).

(a) Let q be a polynomial with integer coefficients and smallest-degree term c_0x^m. Use a property of the Γ-function to show that

$$\frac{1}{m!}\int_0^\infty q(x)e^{-x}\,dx$$

is an integer of the form $c_0 + (m+1)(\text{integer})$.

(b) Let $m > 0$ be an integer. Let $q(x) = x^m(x-1)^{m+1}$, a polynomial having integer coefficients and smallest-degree term $(-1)^{m+1}x^m$. Show that

$$\frac{e}{m!}\int_1^\infty q(x)e^{-x}\,dx = \frac{1}{m!}\int_0^\infty q(x+1)e^{-x}\,dx,$$

and use part (a) to deduce that this integral is an integer multiple of $m+1$.

(c) Let $\varepsilon > 0$ be arbitrary. In the notation of part (b), show that there exists an m such that

$$\frac{e}{m!}\int_0^1 q(x)e^{-x}\,dx < \varepsilon.$$

(d) Suppose $a_0 \neq 0$ and a_1 were integers such that $a_0 + a_1e = 0$. Multiplying through by the integral of $q(x)e^{-x}/m!$, we would have

$$\begin{aligned}0 &= \frac{a_0}{m!}\int_0^\infty q(x)e^{-x}\,dx + \frac{a_1e}{m!}\int_0^\infty q(x)e^{-x}\,dx\\ &= \frac{a_0}{m!}\int_0^\infty q(x)e^{-x}\,dx + \frac{a_1e}{m!}\int_0^1 q(x)e^{-x}\,dx + \frac{a_1e}{m!}\int_1^\infty q(x)e^{-x}\,dx.\end{aligned}$$

By part (b), the third summand is an integer multiple of $m+1$.

Show that for m sufficiently large, the first summand would be an integer not divisible by $m+1$ by part (a), and the second summand would be smaller than $1/2$ by part (c). This is impossible, so our assumption that $a_0 + a_1e = 0$ (with $a_0 \neq 0$) was incorrect.

Exercise 10.29. This argument, adapted from Klein's Evanston Colloquium, generalizes the preceding exercise, showing that e is not the root of a polynomial equation with integer coefficients.

Suppose there exist integers $a_0 \neq 0$, a_1, …, a_n such that

$$(*) \qquad 0 = a_0 + a_1 e + \cdots + a_n e^n = a_0 + \sum_{k=1}^{n} a_k e^k.$$

Throughout, k denotes an integer such that $1 \leq k \leq n$.

(a) Let $m > 0$ be an integer, and consider the polynomial

$$q(x) = x^m\big((x-1)(x-2)\dots(x-n)\big)^{m+1}.$$

Show that the smallest-degree term of $q(x)$ is $\big((-1)^m n!\big)^{m+1} x^m$, and that the smallest-degree term of $q(x+k)$ has degree $m+1$.

(b) Use (a) to show $\dfrac{e^k}{m!}\displaystyle\int_k^\infty q(x)e^{-x}\,dx$ is an integer divisible by $m+1$.

(c) Let $\varepsilon > 0$ be arbitrary. Prove there exists an m such that

$$\frac{e^k}{m!}\int_0^k q(x)e^{-x}\,dx < \varepsilon.$$

Hint: There is an $M > 0$ such that $\big|(x-1)(x-2)\dots(x-n)\big| \leq M$ for $0 \leq x \leq k$ (why?), so $e^k|q(x)e^{-x}| \leq e^k k^m M^{m+1}$.

(d) If (*) holds, then

$$\begin{aligned} 0 &= \frac{1}{m!}\left(a_0 + \sum_{k=1}^{n} a_k e^k\right)\int_0^\infty q(x)e^{-x}\,dx \\ &= \frac{a_0}{m!}\int_0^\infty q(x)e^{-x}\,dx + \sum_{k=1}^{n}\frac{a_k e^k}{m!}\int_0^\infty q(x)e^{-x}\,dx. \end{aligned}$$

Obtain a contradiction, if m is sufficiently large, as follows: Each integral in the sum may be decomposed as

$$\frac{a_k e^k}{m!}\int_0^k q(x)e^{-x}\,dx + \frac{a_k e^k}{m!}\int_k^\infty q(x)e^{-x}\,dx.$$

By part (c), the first integral is smaller than $1/(2n)$ for sufficiently large m (so the sum over k is smaller than $1/2$), and by part (b) the second is an integer multiple of $m+1$. But for m sufficiently large, the "a_0" summand is not a multiple of $m+1$.

Chapter 11

Circular Functions

Trigonometric functions are usually introduced via geometry, either as ratios of sides in a right triangle, or in terms of points on the unit circle. Our analytic development loosely follows Ahlfors. The primary advantage over a geometric approach is that definitions are based on the axioms of $\mathbf{R}$. Fringe benefits include easy generalization to complex variables, and illustration of solving differential equations by power series.

11.1 Sine and Cosine

Theorem 11.1. *There exist unique twice-differentiable functions C and $S : \mathbf{R} \to \mathbf{R}$ satisfying*

$$\begin{aligned} C'' + C &= 0, \qquad C(0) = 1, \quad C'(0) = 0; \\ S'' + S &= 0, \qquad S(0) = 0, \quad S'(0) = 1. \end{aligned}$$

Definition 11.2. The function C in Theorem 11.1 is called the *cosine* function, $\cos : \mathbf{R} \to \mathbf{R}$. The function S is called the *sine* function $\sin : \mathbf{R} \to \mathbf{R}$.

Proof of Theorem 11.1. (Existence). Suppose provisionally that there exists a real-analytic solution y of the second-order differential equation $y'' + y = 0$ in some neighborhood of 0. Write

$$y(x) = \sum_{k=0}^{\infty} a_k x^k = a_0 + a_1 x + a_2 x^2 + \dots,$$

and use term by term differentiation to deduce that

$$\begin{aligned} y'(x) &= \sum_{k=0}^{\infty}(k+1)a_{k+1}x^k, \\ y''(x) &= \sum_{k=0}^{\infty}(k+2)(k+1)a_{k+2}x^k. \end{aligned}$$

By assumption, y satisfies

$$\begin{aligned} 0 = y'' + y &= \sum_{k=0}^{\infty}(k+2)(k+1)a_{k+2}x^k + \sum_{k=0}^{\infty} a_k x^k \\ &= \sum_{k=0}^{\infty}\big[(k+2)(k+1)a_{k+2} + a_k\big]x^k. \end{aligned}$$

The coefficients of this series must all be zero by the identity theorem for power series (Theorem 6.45), so we get a recursion relation

$$a_{k+2} = -\frac{a_k}{(k+2)(k+1)} \quad \text{for all } k \geq 0.$$

The initial conditions $y(0) = a_0$ and $y'(0) = a_1$ determine the first two coefficients. The first two coefficients, together with the recursion relation, determine all the remaining coefficients:

$$a_{2k} = \frac{(-1)^k a_0}{(2k)!}, \qquad a_{2k+1} = \frac{(-1)^k a_1}{(2k+1)!}.$$

The respective initial conditions for C ($a_0 = 1$ and $a_1 = 0$) and S ($a_0 = 0$ and $a_1 = 1$) give

$$\begin{aligned} C(x) &= \sum_{k=0}^{\infty}(-1)^k \frac{x^{2k}}{(2k)!} &&= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots, \\ S(x) &= \sum_{k=0}^{\infty}(-1)^k \frac{x^{2k+1}}{(2k+1)!} &&= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots. \end{aligned}$$

To determine the interval of convergence for each series, apply the ratio test. For $C(x)$, we have

$$\lim_{k\to\infty}\left|\frac{x^{2k+2}/(2k+2)!}{x^{2k}/(2k)!}\right| = \lim_{k\to\infty}\frac{|x|^2\,(2k)!}{(2k+2)!} = \lim_{k\to\infty}\frac{|x|^2}{(2k+2)(2k+1)},$$

which is 0 for all real x; that is, the series for $C(x)$ converges absolutely for every real x. The calculation for $S(x)$ is similar.

In summary, there exist real-analytic functions C and S satisfying the conditions of Theorem 11.1.

(Uniqueness). The existence proof implicitly showed that the conditions of Theorem 11.1 uniquely define *real-analytic* functions C and S. However, there could conceivably be non-analytic solutions.

Lemma 11.3. *Let $y : \mathbf{R} \to \mathbf{R}$ be a twice-differentiable function satisfying $y'' + y = 0$ on $\mathbf{R}$. If $y(0) = y'(0) = 0$, then $y(x) = 0$ for all x.*

Proof. If $y'' + y = 0$ for all real x, then

$$\big((y')^2 + y^2\big)' = 2y'y'' + 2yy' = 2y' \cdot (y'' + y) = 0.$$

By the identity theorem, the function $(y')^2 + y^2$ is constant:

$$y'(x)^2 + y(x)^2 = y'(0)^2 + y(0)^2 = 0 \quad \text{for all real } x.$$

In particular, $y(x) = 0$ for all real x. □

We now complete the proof of uniqueness. Let a and b be arbitrary real numbers, and assume f is any twice-differentiable function satisfying $f'' + f = 0$, $f(0) = a$ and $f'(0) = b$. The function

$$y(x) = f(x) - \big(aC(x) + bS(x)\big)$$

is easily shown to satisfy $y'' + y = 0$, $y(0) = 0$, and $y'(0) = 0$. By the lemma, $y(x) = 0$ for all real x, i.e., $f(x) = aC(x) + bS(x)$.

That is, the differential equation $f'' + f = 0$ and the conditions $f(0) = a$ and $f'(0) = b$ uniquely determine f. This completes the proof of Theorem 11.1. □

For later use, we record the final step of the proof:

Proposition 11.4. *Let a and b be real numbers. If f is a function satisfying $f'' + f = 0$, $f(0) = a$ and $f'(0) = b$, then*

$$f(x) = a\cos x + b\sin x \quad \textit{for all real } x.$$

Theorem 11.5. *The sine and cosine functions satisfy the following identities for all real x, y:*

(i) $\cos(-x) = \cos x, \quad \sin(-x) = -\sin x.$

(ii) $\sin'(x) = \cos(x), \quad \cos'(x) = -\sin(x)$.

(iii) $\sin^2 x + \cos^2 x = 1$.

(iv) $\cos(x + y) = \cos x \cos y - \sin x \sin y,$
$\sin(x + y) = \sin y \cos x + \sin x \cos y.$

In particular, $\sin(2x) = 2 \sin x \cos x$ *and* $\cos(2x) = \cos^2 x - \sin^2 x$.

Proof. (i) It is apparent that sin is an odd function from its power series; however, a direct proof (in the spirit of the theorem) can be given using Proposition 11.4: The function $f(x) = \sin(-x)$ satisfies $f'' + f = 0$, $f(0) = 0$, and (by the chain rule) $f'(0) = -1$, so $f = -\sin$. Evenness of cos is similar.

(ii) If $f = \sin'$, then $f'' + f = 0$ by differentiating $\sin'' + \sin = 0$. But $f(0) = \sin' 0 = 1$ by definition of sin, and $f'(0) = \sin''(0) = -\sin 0 = 0$. By Proposition 11.4, $\sin' = \cos$. A similar argument shows $\cos' = -\sin$.

(iii) Consider the function $f = \sin^2 + \cos^2$. Part (ii) gives

$$f' = 2 \sin \sin' + 2 \cos \cos' = 2 \sin \cos + 2 \cos(-\sin) = 0,$$

i.e., f is constant. Thus $f(x) = f(0) = \sin^2 0 + \cos^2 0 = 1$ for all x.

(iv) To prove the addition formulas, fix a real number y and consider the function $f(x) = \sin(x+y)$. The chain rule gives $f'' + f = 0$, and the derivative formula for sin implies $f'(x) = \cos(x + y)$ for all x. Substituting $x = 0$, we find $f(0) = \sin y$ and $f'(0) = \cos y$. Proposition 11.4 with $a = \sin y$ and $b = \cos y$ gives

$$\sin(x + y) = f(x) = \sin y \cos x + \sin x \cos y$$

The addition formula for cos is similar. ☐

Several auxiliary formulas are listed here for reference. These need not be memorized, but their technique of proof is worth remembering.

Corollary 11.6. *For all real x and y,*

$$\begin{aligned}
\cos x \cos y &= \tfrac{1}{2}\big(\cos(x - y) + \cos(x + y)\big), \\
\sin x \sin y &= \tfrac{1}{2}\big(\cos(x - y) - \cos(x + y)\big), \\
\sin x \cos y &= \tfrac{1}{2}\big(\sin(x + y) + \sin(x - y)\big); \\
\cos x + \cos y &= 2 \cos\big(\tfrac{1}{2}(x + y)\big) \cos\big(\tfrac{1}{2}(x - y)\big), \\
\cos x - \cos y &= 2 \sin\big(\tfrac{1}{2}(x + y)\big) \sin\big(\tfrac{1}{2}(x - y)\big), \\
\sin x + \sin y &= 2 \sin\big(\tfrac{1}{2}(x + y)\big) \cos\big(\tfrac{1}{2}(x - y)\big).
\end{aligned}$$

Proof. From the addition theorem for cos and the fact that sin is odd,

$$\begin{aligned}\cos(x-y) &= \cos x\cos y + \sin x\sin y,\\ \cos(x+y) &= \cos x\cos y - \sin x\sin y.\end{aligned}$$

Adding and subtracting these (and dividing by 2) gives the first two formulas. The third is derived similarly from the addition theorem for sin.

To derive the fourth, write the first as

$$\cos(u+v) + \cos(u-v) = 2\cos u\cos v.$$

Put $u = \frac{1}{2}(x+y)$ and $v = \frac{1}{2}(x-y)$, noting that $x = u+v$ and $y = u-v$. The fifth and sixth formulas are similar. $\square$

Periodicity

Proposition 11.7. *There exists a unique real number α, with*

$$\sqrt{2} < \alpha < \sqrt{6-\sqrt{12}},$$

such that $\cos x > 0$ *for* $-\alpha < x < \alpha$ *and* $\cos\alpha = 0$.

Proof. To prove there exists a smallest positive root α of the cosine function, we establish the estimates

$$\begin{aligned} x - \frac{x^3}{3!} \le \sin x \le x \quad &\text{for all real } x \ge 0,\\ 1 - \frac{x^2}{2!} \le \cos x \le 1 - \frac{x^2}{2!} + \frac{x^4}{4!} \quad &\text{for all real } x.\end{aligned}$$

The identity $\sin^2 t + \cos^2 t = 1$ implies $-1 \le \cos t \le 1$ for all real t. Integrating from 0 to x, with $x \ge 0$ arbitrary, gives

$$-x \le \int_0^x \cos t\,dt = \sin t\Big|_{t=0}^{x} = \sin x \le x.$$

Since $x \ge 0$ was arbitrary, $-t \le \sin t \le t$ for all $t \ge 0$. Integrating *this* from 0 to x gives

$$-\frac{x^2}{2} \le \int_0^x \sin t\,dt = 1 - \cos x \le \frac{x^2}{2},$$

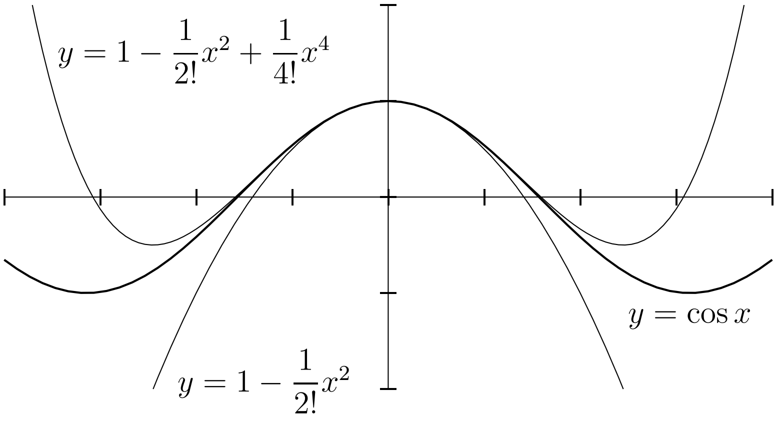
and since $x \geq 0$ was arbitrary it follows that $1 - (t^2/2) \leq \cos t \leq 1$ for $t \geq 0$. Another integration (and renaming of variable) gives

$$x - \frac{x^3}{6} \leq \sin x \leq x \quad \text{for } x \geq 0,$$

and a fourth gives $x^2/2 - x^4/24 \leq 1 - \cos x \leq x^2/2$, or

$$1 - \frac{x^2}{2} \leq \cos x \leq 1 - \frac{x^2}{2} + \frac{x^4}{24} \quad \text{for } x \geq 0.$$

This inequality is true for $x < 0$ because each term is even in x.



Put $p_2(x) = 1 - (x^2/2)$ and $p_4(x) = 1 - (x^2/2) + (x^4/24)$. Clearly $p_2(x) > 0$ for $0 < x < \sqrt{2}$, and by the quadratic formula, $p_4(x) > 0$ for $0 < x < \sqrt{6 - \sqrt{12}}$. The preceding inequality for cos implies

$$\cos(\sqrt{6 - \sqrt{12}}) \leq 0 \leq \cos \sqrt{2}.$$

The intermediate value theorem guarantees that cos has a zero between $\sqrt{2} \approx 1.41421$ and $\sqrt{6 - \sqrt{12}} \approx 1.59245$. This is the unique zero in $[0, \sqrt{6}]$, because

$$\cos' x = -\sin x \leq -x + \frac{x^3}{6} < 0 \quad \text{for } 0 < x < \sqrt{6}.$$

Evenness of cos implies $0 < \cos x$ for $-\alpha < x < \alpha$. □

Proposition 11.8. *If $\alpha > 0$ is the smallest positive root of* cos*, then*

$$\cos(x + 4\alpha) = \cos x, \qquad \sin(x + 4\alpha) = \sin x, \quad \textit{for all real } x.$$

Definition 11.9. The smallest positive period of the cosine function is called 2π; that is, $\pi = 2\alpha$.

Remark 11.10. The number π is the familiar ratio of the circumference of a circle to its diameter, see below. Proposition 11.7 guarantees that

$$2.82842 \le \pi \le 3.1849.$$

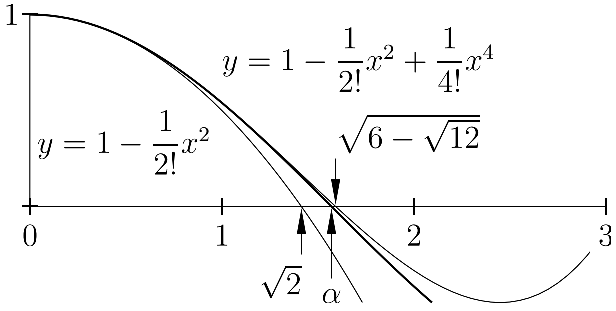


Figure 11.1: The smallest positive zero of the cosine function.

Proof. Since $\cos(\pi/2) = 0$, we have $\sin(\pi/2) = \pm 1$. The proof of Proposition 11.7 guarantees that $0 \le x - x^3/6 \le \sin x$ for $0 \le x \le \sqrt{6}$, so we have $\sin(\pi/2) = 1$. The addition formulas for sin and cos give

$$\begin{aligned}\sin(x + \pi/2) &= \sin x \cos(\pi/2) + \cos x \sin(\pi/2) = \quad \cos x,\\ \cos(x + \pi/2) &= \cos x \cos(\pi/2) - \sin x \sin(\pi/2) = -\sin x.\end{aligned}$$

Bootstrapping, we have $\sin(x+\pi) = \cos(x+\pi/2) = -\sin x$, and therefore $\sin(x+2\pi) = -\sin(x+\pi) = \sin x$ for all real x. The cosine function is 2π-periodic as well since $\cos x = \sin(x + \pi/2)$ for all x. □

Remark 11.11. The proof shows sin and cos are π-*anti-periodic*, in the sense that

$$\sin(x + \pi) = -\sin x, \qquad \cos(x + \pi) = -\cos x \quad \text{for all real } x.$$

Consequently, $\sin x = 0$ if and only if $x = k\pi$ for some integer k, while $\cos x = 0$ if and only if $x = (k + \frac{1}{2})\pi$ for some integer k.

Theorem 11.12. *For all real x, $\cos x = \sin(\pi/2 - x)$. Further,*

$$\begin{aligned}\sin(\pi/4) = \cos(\pi/4) &= \sqrt{2}/2,\\ \sin(\pi/6) = \cos(\pi/3) &= 1/2,\\ \cos(\pi/6) = \sin(\pi/3) &= \sqrt{3}/2.\end{aligned}$$

Proof. The identity $\cos x = \sin(\pi/2 - x)$ follows from the addition formula for sin and evenness of cos. Taking $x = \pi/4$ shows that $\cos(\pi/4) = \sin(\pi/4)$; since each is positive and the sum of their squares is 1, each is equal to $\sqrt{1/2} = \sqrt{2}/2$.

Repeated use of the addition formulas gives, for all real x,

$$\begin{aligned}\cos 3x &= \cos(2x + x) = \cos(2x)\cos x - \sin(2x)\sin x \\ &= (\cos^2 x - \sin^2 x)\cos x - (2\sin x\cos x)\sin x \\ &= \cos x(\cos^2 x - 3\sin^2 x) = \cos x(1 - 4\sin^2 x).\end{aligned}$$

Let $x = \pi/6$. Since $\cos 3x = \cos(\pi/2) = 0$ but $\cos x > 0$, we have $1 = 4\sin^2 x$. But $\sin x > 0$ since $0 < x < \pi/2$. We conclude that $\sin(\pi/6) = 1/2$, and therefore that $\cos(\pi/3) = \sin(\pi/6) = 1/2$.

If $0 \leq x \leq \pi/2$, then $\cos x = \sqrt{1 - \sin^2 x}$ since both cos and sin are non-negative on this interval. The preceding conclusions immediately imply $\cos(\pi/6) = \sin(\pi/3) = \sqrt{3}/2$. ☐

The Unit Circle

Theorem 11.13. *Let x and y be real numbers satisfying $x^2 + y^2 = 1$. There exists a unique real number θ, $0 \leq \theta < 2\pi$, such that $x = \cos\theta$ and $y = \sin\theta$.*

Proof. Since cos is continuous and strictly decreasing on $[0, \pi]$, and $\cos 0 = 1$, $\cos\pi = -1$, Theorem 8.33 implies cos maps $[0, \pi]$ bijectively to $[-1, 1]$. A similar argument shows cos maps $(\pi, 2\pi)$ bijectively to $(-1, 1)$.

If a point (x, y) on the circle $x^2 + y^2 = 1$ lies in the closed upper half-plane, i.e., if $0 \leq y \leq 1$, then $y = \sqrt{1 - x^2}$. By the preceding paragraph there exists a unique θ in $[0, \pi]$ with $x = \cos\theta$. Since $\sin\theta \geq 0$ for $0 \leq \theta \leq \pi$, the point $(\cos\theta, \sin\theta)$ is equal to (x, y).

If instead (x, y) lies in the open lower half-plane, i.e., $-1 \leq y < 0$, then $y = -\sqrt{1 - x^2}$ and $-1 < x < 1$. There is a unique θ in $(\pi, 2\pi)$ such that $x = \cos\theta$. Since $\sin\theta < 0$ for $\pi < \theta < 2\pi$, the point $(\cos\theta, \sin\theta)$ is equal to (x, y).

In summary, the mapping $\theta \mapsto (\cos\theta, \sin\theta)$ is a bijection from the half-open interval $[0, 2\pi) = [0, \pi] \cup (\pi, 2\pi)$ to the unit circle. ☐

Remark 11.14. If a particle has position $(\cos\theta, \sin\theta)$ for $0 \leq \theta \leq 2\pi$, the preceding argument shows that the particle traces the unit circle once. The particle's speed at time θ is $\sqrt{(\cos'\theta)^2 + (\sin'\theta)^2} = 1$, so the distance traveled, a.k.a., the circumference of the unit circle, is 2π.

11.2 Auxiliary Trig Functions

Definition 11.15. On the set of real numbers x such that $\cos x \neq 0$, define the *secant* function $\sec x = 1/\cos x$.

Remark 11.16. The secant is even, π-anti-periodic, and $|\sec x| \geq 1$ for all x in the domain.

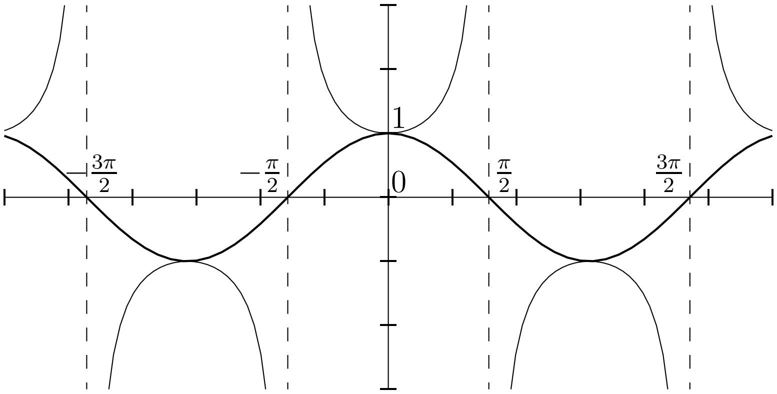


Figure 11.2: The cosine and secant functions. (Tick marks are integers.)

Definition 11.17. On the set of real numbers x such that $\sin x \neq 0$, define the *cosecant* function $\csc x = 1/\sin x$.

Remark 11.18. The cosecant function is odd, π-anti-periodic, and satisfies the identity $\csc(x + \frac{\pi}{2}) = \sec x$.

Definition 11.19. On the set of real numbers x such that $\cos x \neq 0$, define the *tangent* function, $\tan x = \sin x/\cos x$.

Remark 11.20. The tangent function is odd, as a quotient of an odd function by an even function, and periodic with period π:

$$\tan(x + \pi) = \frac{\sin(x + \pi)}{\cos(x + \pi)} = \frac{-\sin x}{-\cos x} = \tan x.$$

Proposition 11.21. *The tangent function is differentiable on the "period interval" $(-\pi/2, \pi/2)$, satisfies $\tan' x = \sec^2 x$ for all x in the domain, and maps the period interval bijectively to* $\mathbf{R}$.

For all real x and y,

$$\tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}.$$

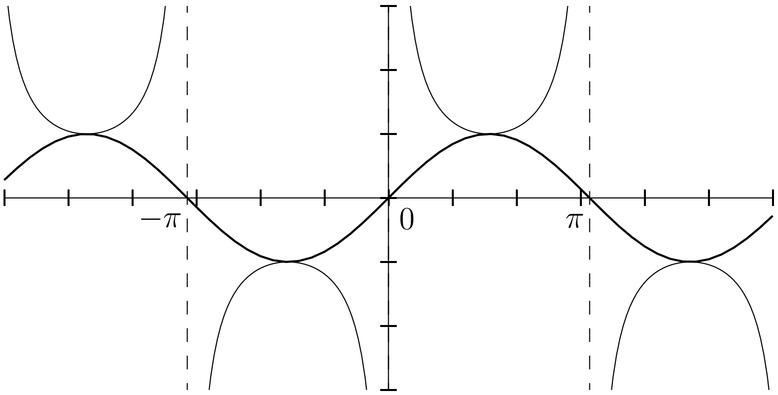


Figure 11.3: The sine and cosecant functions. (Tick marks are integers.)

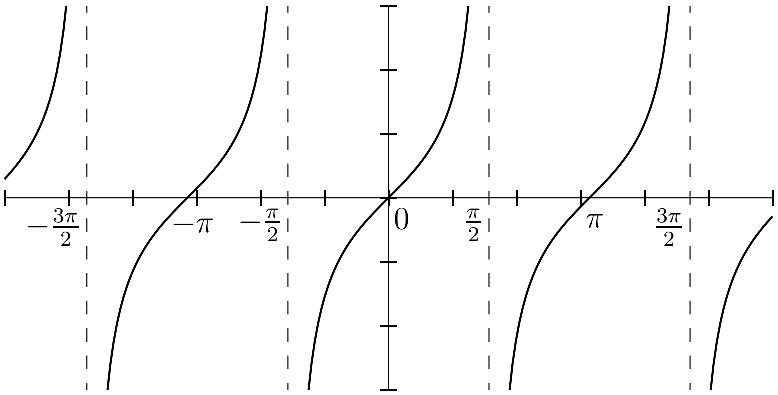


Figure 11.4: The tangent function. (Tick marks are integers.)

Proof. The quotient rule gives

$$\tan' = \frac{\cos\sin' - \sin\cos'}{\cos^2} = \frac{\cos^2 + \sin^2}{\cos^2} = \frac{1}{\cos^2} = \sec^2 \geq 1.$$

In particular, tan is strictly increasing (hence injective) on $(-\pi/2, \pi/2)$. Since cos and sin are positive on $(0, \pi/2)$, $\tan x = \sin x/\cos x \to \infty$ as $x \to \pi/2^-$. Since tan is odd, $\tan x \to -\infty$ as $x \to -\pi/2^+$. A standard Intermediate Value Theorem argument shows tan maps the period interval surjectively to **R**.

For all real x and y,

$$\tan(x+y) = \frac{\sin(x+y)}{\cos(x+y)} = \frac{\sin x\cos y + \cos x\sin y}{\cos x\cos y - \sin x\sin y} = \frac{\tan x + \tan y}{1 - \tan x\tan y};$$

the final equality follows by dividing the numerator and denominator by $\cos x \cos y$. ☐

Corollary 11.22. *For every integer k, there exists a unique real number x such that $(k - \frac{1}{2})\pi < x < (k + \frac{1}{2})\pi$ and $\tan x = x$.*

Proof. The function $f(x) = \tan x - x$ is differentiable, with derivative $f'(x) = \sec^2 x - 1 \geq 0$. Since the derivative vanishes at isolated points, namely at $x = k\pi$, f is strictly increasing on $\big((k - \frac{1}{2})\pi, (k + \frac{1}{2})\pi\big)$. On this interval, x is bounded while tan approaches $-\infty$ at the left endpoint and approaches ∞ at the right endpoint. By the Intermediate Value Theorem, f maps $\big((k-\frac{1}{2})\pi, (k+\frac{1}{2})\pi\big)$ bijectively to $\mathbf{R}$, and in particular is equal to 0 exactly once in this interval. ☐

Definition 11.23. For x not an integer multiple of π, define the *cotangent* function $\cot x = \cos x / \sin x$.

Remark 11.24. The cotangent is periodic with period π. Away from integer multiples of $\pi/2$, $\tan = \sin / \cos$ and $\cot = \cos / \sin$ are reciprocals, and $\cot(x + \frac{\pi}{2}) = -\tan x$. The derivative of cot is $-1/\csc^2$, so cot is decreasing on every interval of the form $\big(k\pi, (k+1)\pi\big)$ with k an integer.

11.3 Inverse Trig Functions

Each circular function is periodic, hence has no "global" inverse. Instead, we restrict each function to an interval on which the function is injective, obtaining a branch of inverse.

Inverse Sine and Cosine

Definition 11.25. The restriction of sin to the interval $[-\pi/2, \pi/2]$ is denoted Sin. The inverse function $\mathrm{Sin}^{-1} : [-1, 1] \to [-\pi/2, \pi/2]$, sometimes denoted arcsin, is called the *principal branch of arcsin.*

Remark 11.26. By definition,

$$\begin{aligned} \sin(\mathrm{Sin}^{-1} x) &= x && \text{for all } x \text{ in } [-1, 1], \\ \mathrm{Sin}^{-1}(\sin y) &= y && \text{for all } y \text{ in } [-\pi/2, \pi/2]. \end{aligned}$$

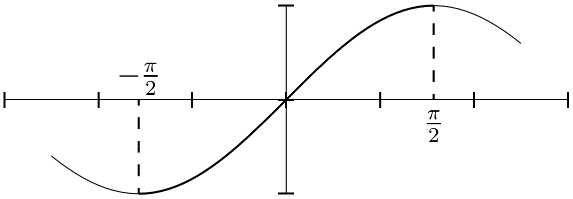


Figure 11.5: Sin.

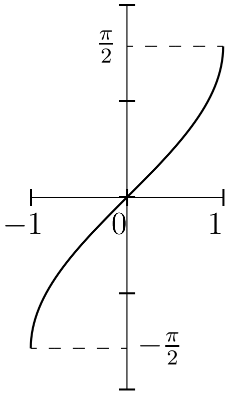


Figure 11.6: Sin^{-1}.

Proposition 11.27. *The function* Sin^{-1} *is continuous on* $[-1, 1]$*, differentiable on* $(-1, 1)$*, and*

$$(\mathrm{Sin}^{-1})'(x) = \frac{1}{\sqrt{1-x^2}} \quad \textit{for } -1 < x < 1.$$

Proof. By Theorem 8.27, Sin^{-1} is differentiable since $\sin' y = \cos y \neq 0$ for $-\pi/2 < y < \pi/2$, and

$$(\mathrm{Sin}^{-1})'(x) = \frac{1}{\cos(\mathrm{Sin}^{-1} x)} = \frac{1}{\sqrt{1-\sin^2(\mathrm{Sin}^{-1} x)}} = \frac{1}{\sqrt{1-x^2}}. \qquad \square$$

Corollary 11.28. *For* $a > 0$*,* $\displaystyle\int_0^a \frac{dx}{\sqrt{a^2-x^2}} = \frac{\pi}{2}$.

Proof. For $0 \leq x < a$,

$$\frac{1}{\sqrt{a^2-x^2}} = \frac{1}{\sqrt{(a+x)(a-x)}} \leq \frac{1}{\sqrt{a(a-x)}},$$

and the upper bound is improperly integrable by Proposition 9.15. The smaller improper integral therefore converges. Factoring a^2 from the radicand and using the substitution $u = x/a$ gives

$$\int_0^a \frac{dx}{\sqrt{a^2-x^2}} = \int_0^a \frac{dx/a}{\sqrt{1-(x/a)^2}} = \int_0^1 \frac{dx}{\sqrt{1-x^2}}.$$

By the second fundamental theorem, this is $\mathrm{Sin}^{-1} 1 - \mathrm{Sin}^{-1} 0 = \dfrac{\pi}{2}$. $\square$

Definition 11.29. The restriction of cos to the interval $[0, \pi]$ is denoted Cos. The inverse function $\mathrm{Cos}^{-1} : [-1, 1] \to [0, \pi]$, sometimes denoted arccos, is called the *principal branch of arccos*.

Remark 11.30. The identity $\cos y = x = \sin(\pi/2 - y)$ for $0 \leq y \leq \pi$ implies $\mathrm{Cos}^{-1} x = y = (\pi/2) - \mathrm{Sin}^{-1} x$ for x in $[-1, 1]$.

Inverse Tangent

Definition 11.31. The inverse of tan on the interval $(-\pi/2, \pi/2)$ is the *principal branch of arctan*, denoted $\mathrm{Tan}^{-1} : \mathbf{R} \to (-\pi/2, \pi/2)$.

Remark 11.32. By definition,

$$\begin{aligned} \tan(\mathrm{Tan}^{-1} x) = x &\quad \text{for all real } x, \\ \mathrm{Tan}^{-1}(\tan y) = y &\quad \text{for all } y \text{ in } (-\pi/2, \pi/2). \end{aligned}$$

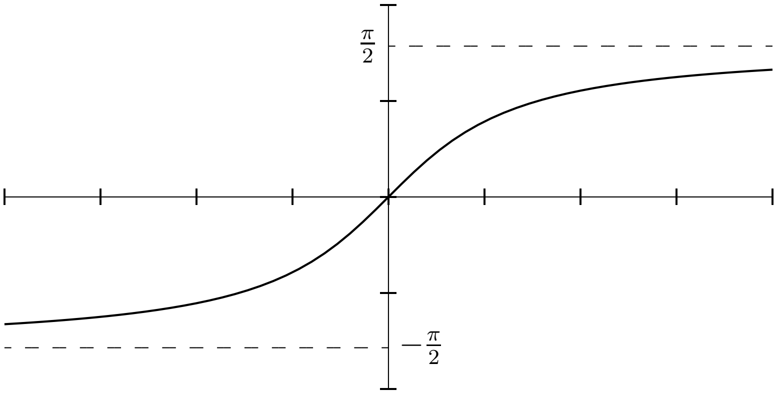


Figure 11.7: The principal branch of arctan.

Proposition 11.33. *The function* Tan^{-1} *is differentiable on* $\mathbf{R}$*, and*

$$(\mathrm{Tan}^{-1})'(x) = \frac{1}{1 + x^2} \quad \textit{for all real } x.$$

Proof. Since $\tan' x = \sec^2 x > 0$ for all real x, Theorem 8.27 guarantees that Tan^{-1} is differentiable, and

$$(\mathrm{Tan}^{-1})'(x) = \frac{1}{\sec^2(\mathrm{Tan}^{-1} x)} = \frac{1}{1 + \tan^2(\mathrm{Tan}^{-1} x)} = \frac{1}{1 + x^2}. \qquad \square$$

Corollary 11.34. *For every* $a > 0$, $\displaystyle\int_0^\infty \frac{dx}{a^2 + x^2} = \frac{\pi}{2a}$.

Proof. The integrand is non-negative and bounded above by $1/x^2$, so the improper integral converges by Proposition 9.15. If $a = 1$, the second fundamental theorem gives

$$\int_0^\infty \frac{dx}{1 + x^2} = \lim_{b\to\infty} \int_0^b \frac{dx}{1 + x^2} = \lim_{b\to\infty} \operatorname{Tan}^{-1} b - \operatorname{Tan}^{-1} 0 = \frac{\pi}{2}.$$

If $a \neq 1$, use the substitution $u = x/a$:

$$\int_0^\infty \frac{dx}{a^2 + x^2} = \int_0^\infty \frac{a \cdot dx/a}{a^2\bigl[1 + (x/a)^2\bigr]} = \frac{1}{a}\int_0^\infty \frac{du}{1 + u^2} = \frac{\pi}{2a}. \qquad \square$$

Corollary 11.35. *For* $-1 < x < 1$, $\operatorname{Tan}^{-1} x$ *may be expanded in a power series*

$$\operatorname{Tan}^{-1} x = \sum_{k=0}^\infty (-1)^k \frac{x^{2k+1}}{2k+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots.$$

Proof. The geometric series

$$\frac{1}{1 + t^2} = \sum_{k=0}^\infty (-1)^k t^{2k} = 1 - t^2 + t^4 - t^6 + \dots$$

converges absolutely for all t in $(-1, 1)$. Integrating term by term,

$$\begin{aligned}\operatorname{Tan}^{-1} x = \int_0^x \frac{dt}{1 + t^2} &= \int_0^x \sum_{k=0}^\infty (-1)^k t^{2k}\,dt \\ &= \sum_{k=0}^\infty (-1)^k \int_0^x t^{2k}\,dt = \sum_{k=0}^\infty (-1)^k \frac{x^{2k+1}}{2k+1}. \qquad \square\end{aligned}$$

Corollary 11.36. $\displaystyle\pi = 2\sqrt{3}\sum_{k=0}^\infty (-1)^k \frac{1}{3^k(2k+1)}$.

Proof. By Theorem 11.12, $\tan(\pi/6) = \sin(\pi/6)/\cos(\pi/6) = 1/\sqrt{3}$, so $\pi = 6\operatorname{Tan}^{-1}(1/\sqrt{3})$. Setting $x = 1/\sqrt{3}$ in the power series for Tan^{-1} and noting that $x^{2k+1} = 1/(3^k\sqrt{3})$ and $6/\sqrt{3} = 2\sqrt{3}$,

$$\pi = 6\sum_{k=0}^\infty (-1)^k \frac{1}{3^k\sqrt{3}(2k+1)} = 2\sqrt{3}\sum_{k=0}^\infty (-1)^k \frac{1}{3^k(2k+1)}. \qquad \square$$

Remark 11.37. This series alternates, so "the tail is bounded by the size of the first omitted term", i.e.,

$$\left|\frac{\pi}{\sqrt{3}} - 2\sum_{k=0}^{n}(-1)^k \frac{1}{3^k(2k+1)}\right| \leq \frac{2}{3^{n+1}(2n+3)}$$

For example, taking $n = 10$ (i.e., 11 terms) gives a rational estimate of $\pi/\sqrt{3}$ that is accurate to within $\varepsilon = 2/4,074,382 < 0.5 \times 10^{-6}$.

11.4 Applications

Theorem 11.38. *Let $n \geq 1$ be an integer. For all real numbers a and b,*

$$\begin{aligned}\int_a^b \cos^n x\,dx &= \frac{1}{n}\cos^{n-1} x \sin x\Big|_{x=a}^{x=b} + \frac{n-1}{n}\int_a^b \cos^{n-2} x\,dx,\\ \int_a^b \sin^n x\,dx &= -\frac{1}{n}\sin^{n-1} x \cos x\Big|_{x=a}^{x=b} + \frac{n-1}{n}\int_a^b \sin^{n-2} x\,dx,\end{aligned}$$

Proof. Integrating by parts with

$$\begin{aligned} u &= \cos^{n-1} x, & v &= \sin x\,dx,\\ du &= -(n-1)\cos^{n-2} x \sin x\,dx, & dv &= \cos x\,dx,\end{aligned}$$

and using $\sin^2 x = 1 - \cos^2 x$ gives

$$\begin{aligned}\int_a^b \cos^n x\,dx &= \cos^{n-1} x \sin x\Big|_{x=a}^{x=b} + (n-1)\int_a^b \cos^{n-2} x \sin^2 x\,dx,\\ &= \cos^{n-1} x \sin x\Big|_{x=a}^{x=b} + (n-1)\int_a^b \cos^{n-2} x\,dx\\ &\qquad - (n-1)\int_a^b \cos^n x\,dx.\end{aligned}$$

Moving the last integral to the left-hand side and dividing by n gives the first formula. The second formula is left as an exercise. □

Corollary 11.39. *If $n \geq 0$ is an integer and*

$$I(n) = \int_0^{\pi/2} \cos^n x\,dx = \int_0^{\pi/2} \sin^n x\,dx,$$

then

$$I(2k) = \frac{(2k-1)}{(2k)} \cdot \frac{(2k-3)}{(2k-2)} \cdots \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \frac{(2k)!}{(2^k \cdot k!)^2} \cdot \frac{\pi}{2},$$
$$I(2k+1) = \frac{(2k)}{(2k+1)} \cdot \frac{(2k-2)}{(2k-1)} \cdots \frac{6}{7} \cdot \frac{4}{5} \cdot \frac{2}{3} = \frac{(2^k \cdot k!)^2}{(2k+1)!}.$$

Proof. Applying the theorem with $a = 0$ and $b = \pi/2$, the boundary terms vanish, so

$$I(n) = \frac{n-1}{n} \cdot I(n-2), \qquad I(0) = \frac{\pi}{2}, \quad I(1) = 1.$$

The "expanded" forms of the integrals follow immediately. To express these in terms of factorials, note that

$$(2k)\cdot(2k-2)\dots 6\cdot 4\cdot 2 = 2^k\cdot k!, \quad (2k-1)\cdot(2k-3)\dots 5\cdot 3\cdot 1 = \frac{(2k)!}{2^k \cdot k!}. \ \square$$

Definition 11.40. The β *function* is defined by

$$\beta(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1}\, dt \quad \text{for } x,\ y > 0.$$

Theorem 11.41. $\beta(x, y) = \dfrac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$ *for all* x, $y > 0$.

Proof. Direct calculation gives $\beta(1, y) = 1/y$ for $y > 0$. Further, $\log\beta(\cdot, y)$ is convex in the sense that if $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\beta\big(\tfrac{x}{p} + \tfrac{z}{q}, y\big) \le \beta(x, y)^{1/p}\beta(z, y)^{1/q} \quad \text{for all } x,\ y,\ z > 0.$$

The proof is entirely similar to log convexity of Γ. Finally,

$$\beta(x+1, y) = \int_0^1 t^x(1-t)^{y-1}\, dt.$$

Integrating by parts, using $u = t^x$ and $dv = (1-t)^{y-1}\, dt$, gives

$$\beta(x+1, y) = -\frac{1}{y}t^x(1-t)^y\bigg|_{t=0}^{t=1} + \frac{x}{y}\int_0^1 t^{x-1}(1-t)^y\, dt.$$

The boundary term is zero, and since

$$\begin{aligned} t^{x-1}(1-t)^y &= t^{x-1}(1-t)^{y-1}(1-t) \\ &= t^{x-1}(1-t)^{y-1} - t^x(1-t)^{y-1}, \end{aligned}$$

the integral is equal to

$$\frac{x}{y}\left[\int_0^1 t^{x-1}(1-t)^{y-1}\,dt - \int_0^1 t^{x}(1-t)^{y-1}\,dt\right] = \frac{x}{y}\big[\beta(x,y)-\beta(x+1,y)\big].$$

Rearranging, $\beta(x+1,y) = \frac{x}{x+y}\beta(x,y)$. Putting these pieces together, the function

$$f(x) = \frac{\Gamma(x+y)}{\Gamma(y)}\beta(x,y)$$

satisfies properties (a)–(c) of Theorem 10.32, so $f = \Gamma$. ☐

Corollary 11.42. $2\displaystyle\int_0^{\pi/2} (\sin\theta)^{2x-1}(\cos\theta)^{2y-1}\,d\theta = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$

Proof. Substitute $t = \sin^2\theta$ in the definition of β. ☐

Corollary 11.43. $\displaystyle\int_{-\infty}^{+\infty} e^{-u^2}\,du = \sqrt{\pi}.$

Proof. Taking $x = y = 1/2$ in Corollary 11.42,

$$\pi = 2\int_0^{\pi/2} d\theta = \frac{\Gamma(\frac{1}{2})^2}{\Gamma(1)}, \quad \text{or} \quad \sqrt{\pi} = \Gamma(\tfrac{1}{2}) = \int_0^\infty t^{-1/2}e^{-t}\,dt.$$

The substitution $u = \sqrt{t}$, $du = dt/2\sqrt{t}$ transforms the last integral into

$$\int_0^\infty e^{-u^2}\,du = \frac{\sqrt{\pi}}{2}.$$

Since the integrand is even, the claim follows. ☐

Remark 11.44. Taking $x = 1/2$ or $y = 1/2$ expresses the integrals of powers of sin and cos in terms of Γ, compare Corollary 11.39.

The Sine Integral

An extended example will demonstrate a variety of techniques developed so far.

Example 11.45. If $x \neq 0$, then

$$s(x) = \frac{\sin x}{x} = \sum_{k=0}^{\infty}(-1)^k\frac{x^{2k}}{(2k+1)!} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \dots.$$

The radius of convergence is ∞, just as for the sine series, so we may view s as a real-analytic function on $\mathbf{R}$ satisfying $s(0) = 1$.

The *sine integral* function is defined by

$$\text{SI}(x) = \int_0^x \frac{\sin t\,dt}{t} = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)(2k+1)!} = x - \frac{x^3}{3 \cdot 3!} + \frac{x^5}{5 \cdot 5!} - \dots.$$

The critical points of SI are the zeros of s, namely the non-zero integer multiples of π. The critical values are, for each $n > 0$,

$$\text{SI}\big((n+1)\pi\big) = \sum_{k=0}^{n} \int_{k\pi}^{(k+1)\pi} \frac{\sin t\,dt}{t} = \sum_{k=0}^{n} a_k, \quad a_k = \int_{k\pi}^{(k+1)\pi} \frac{\sin t\,dt}{t}.$$

The quotient rule gives

$$\text{SI}''(x) = s'(x) = \frac{x\cos x - \sin x}{x^2}, \quad x \neq 0.$$

The derivative $s'(0)$ may be evaluated using the calculus of sloppiness: $\cos x \approx 1 + O(x^2)$ and $\sin x \approx x + O(x^3)$ for $x \approx 0$, so

$$s'(x) = \frac{x - x + O(x^3)}{x^2} \approx O(x) \to 0.$$

The critical points of s are solutions of $x\cos x - \sin x = 0$. At each critical point, $\cos x \neq 0$, since $\sin x = \pm 1$ when $\cos x = 0$. Consequently, $s'(x) = 0$ if and only if $x = \tan x$. By Corollary 11.22, there is exactly one critical point of s in each interval $(k - \frac{1}{2})\pi < x < (k + \frac{1}{2})\pi$.

The function s "oscillates", but the size of the oscillations decreases with $|t|$. Precisely,

$$s(t + 2\pi) = \frac{\sin t}{t + 2\pi} < s(t) \quad \text{for all } t \geq 0.$$

Since

$$\frac{|\sin t|}{(k+1)\pi} \leq \frac{|\sin t|}{t} \leq \frac{|\sin t|}{k\pi} \quad \text{for } k\pi \leq t \leq (k+1)\pi,$$

we have

$$\frac{2}{(k+1)\pi} \leq \left| \int_{k\pi}^{(k+1)\pi} \frac{\sin t}{t}\,dt \right| = |a_k| \leq \frac{2}{k\pi} \quad \text{for } k \geq 0.$$

The signs of the integrals a_k alternate, since sin is alternately positive and negative between integer multiples of π, and $|a_{k+1}| < |a_k|$ for all $k \geq 0$. The absolute maximum value of SI is therefore the first critical value

$$\mathrm{SI}(\pi) = \int_0^\pi \frac{\sin t\, dt}{t}.$$

By the alternating series test, $\lim\bigl(\mathrm{SI}(n\pi), \infty\bigr)$ exists, and since the partial sums bound the partial integrals, the improper integral

$$\lim_{x\to\infty} \mathrm{SI}(x) = \int_0^\infty \frac{\sin t\, dt}{t}$$

converges. However, the integral is not absolutely convergent, since

$$\int_0^\infty \left|\frac{\sin t}{t}\right| dt = \sum_{k=0}^\infty \int_{k\pi}^{(k+1)\pi} \left|\frac{\sin t}{t}\right| dt \geq \sum_{k=0}^\infty \frac{2}{(k+1)\pi}$$

is (a multiple of) the harmonic series.

Exercises

Exercise 11.1. Use termwise differentiation to give an alternative proof that $\sin' = \cos$ and $\cos' = -\sin$.

Exercise 11.2. Use the trigonometric power series to evaluate:

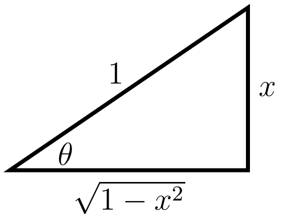
(a) $\displaystyle\lim_{x\to 0} \frac{1-\cos x}{x^2}$. (b) $\displaystyle\lim_{x\to 0} \int_0^{x^2} \frac{1-\cos t}{x^6}\, dt$.

Exercise 11.3. Prove that $\sec' = \sec \cdot \tan$ (including that the domains are the same).

Exercise 11.4. Prove that $\cot 2x = \frac{1}{2}(\cot x - \tan x)$ if x is not an integer multiple of $\pi/2$.

Exercise 11.5. Evaluate $\sin\frac{\pi}{8}$ and $\cos\frac{\pi}{8}$ exactly. (Your answers should involve only square roots and rational numbers.)

Exercise 11.6. Let $0 < x < 1$, and let $\theta = \mathrm{Sin}^{-1} x$:



It follows immediately that $\cos\theta = \sqrt{1-x^2}$,

$$\sec\theta = \frac{1}{\sqrt{1-x^2}}, \quad \tan\theta = \frac{x}{\sqrt{1-x^2}}, \quad \cot\theta = \frac{\sqrt{1-x^2}}{x}.$$

Similarly, find $\cos\operatorname{Tan}^{-1} x$, $\sec\operatorname{Tan}^{-1} x$, and $\sin\operatorname{Tan}^{-1} x$.

Exercise 11.7. Use the substitution $u = x - (1/x)$ to evaluate the improper integral

$$\int_0^\infty \frac{x^2+1}{x^4+1}\,dx.$$

Exercise 11.8. This exercise outlines an evaluation of the improper integral

$$\int_0^\pi \log\sin x\,dx.$$

(a) Show the improper integral converges.
Suggestion: Split the interval at $\pi/2$ and use symmetry to deduce the two integrals are equal. To prove that each converges, first show that $x/2 < \sin x < x$ for $0 < x \leq \pi/2$.

(b) Use the substitution $x = 2u$ and the double angle formula for $\sin 2u$ to obtain an algebraic equation for the integral, and solve.

Exercise 11.9. Prove that the *Fresnel integrals*

$$\int_0^\infty \cos(x^2)\,dx \quad \text{and} \quad \int_0^\infty \sin(x^2)\,dx$$

converge conditionally. (Do not attempt to evaluate.)

Exercise 11.10. Show that the power series

$$f(x) = \sum_{k=0}^\infty (-1)^k \frac{x^k}{(2k)!} = 1 - \frac{x}{2!} + \frac{x^2}{4!} - \frac{x^3}{6!} + \dots$$

has infinite radius, and $f(x^2) = \cos x$ and $f(-x^2) = \cosh x$ for $x \geq 0$.

Exercise 11.11. Using the Lagrange form of the remainder, how large could

$$\left|\cos x - \sum_{k=0}^n (-1)^k \frac{x^{2k}}{(2k)!}\right|$$

be if $|x| \leq \pi/2 < 1.6$? How many terms are sufficient to estimate $\cos x$ on this interval to within $\varepsilon < 0.5 \times 10^{-6}$? $\varepsilon < 0.5 \times 10^{-8}$?

Exercise 11.12. By continuing the estimates in the proof of Proposition 11.7, prove inductively that for all $n \geq 0$ and all real $x \geq 0$,

$$\sum_{k=0}^{2n+1}(-1)^k \frac{x^{2k}}{(2k)!} \leq \cos x \leq \sum_{k=0}^{2n}(-1)^k \frac{x^{2k}}{(2k)!},$$
$$\sum_{k=0}^{2n+1}(-1)^k \frac{x^{2k+1}}{(2k+1)!} \leq \sin x \leq \sum_{k=0}^{2n}(-1)^k \frac{x^{2k+1}}{(2k+1)!}.$$

In words, the Taylor polynomials for cos and sin alternately under-estimate and over-estimate the function values on the positive reals. (The fact that the coefficients alternate in sign is not enough to deduce this result, but *is* a good way of remembering it.)

The following multi-part exercise presents Ivan Niven's proof that π^2 is irrational. It follows that π itself is irrational.

Exercise 11.13. Define $f_n : [0, 1] \to \mathbf{R}$ by

$$f_n(x) = \frac{x^n(1-x)^n}{n!}.$$

Prove each of the following assertions.

(a) If $0 < x < 1$, then $0 < f_n(x) < \frac{1}{n!}$.

(b) The derivatives $f_n^{(k)}(0)$ and $f_n^{(k)}(1)$ are integers for all $k \geq 0$.

Assume $\pi^2 = p/q$ in lowest terms, and let

$$F_n(x) = q^n \sum_{k=0}^{n}(-1)^k f_n^{(2k)}(x)\, \pi^{2n-2k}.$$

(c) $F_n(0)$ and $F_n(1)$ are integers.

(d) $\pi^2 p^n f_n(x) \sin \pi x = \dfrac{d}{dx}\left(F_n'(x) \sin \pi x - \pi F_n(x) \cos \pi x\right)$.

(e) $F_n(1) + F_n(0) = \pi p^n \displaystyle\int_0^1 f_n(x) \sin \pi x \, dx$.

(f) If n is sufficiently large, then $0 < F_n(0) + F_n(1) < 1$.

In short, if π^2 is rational, there exists an integer between 0 and 1.

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